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Fully explicit and super-efficient iterative methods for solving systems of nonlinear equations

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Abstract. In this paper, we propose novel fully explicit and super-efficient iterative methods for solving systems of nonlinear equations. The proposed methods are matrix-free and do not require expensive operations such as matrix inversion or matrix multiplication; instead, they involve only vector operations. As a result, they are easy to implement with $O(n^2)$ computational complexity, whereas other existing iterative methods typically have $O(n^3)$ complexity. Numerical experiments are presented to demonstrate the effectiveness, robustness, and superior performance of the proposed iterative methods.

1. Introduction

In recent years, researchers have devoted considerable attention to constructing higher-order iterative methods for solving systems of nonlinear equations [1–5, 10, 11] and references therein. Among these, the "best iterations" with vector and scalar coefficients were presented in the recent paper [12]. They require only one matrix inversion and have high efficiency; therefore, they are very interesting from the computational point of view. For convenience, we recall some definitions and notations from [4, 6, 13]. Let $a = (a_1, a_2, \dots, a_n)^T \in \mathbb{R}^n$ and $b = (b_1, b_2, \dots, b_n)^T \in \mathbb{R}^n$. The point-wise multiplication and division of two vectors in $\mathbb{R}^n$ are defined as:

$$a \cdot b = (a_1 b_1, a_2 b_2, \dots, a_n b_n)^T \in \mathbb{R}^n, \tag{1.1a}$$

$$\frac{a}{b} = \Big(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\Big)^T \in \mathbb{R}^n. \tag{1.1b}$$

Of course, division of vectors (1.1b) takes place, when $b \neq 0$, which is understood component-wise. From (1.1a) and (1.1b), it immediately follows that

$$\mathbf{1}a = a, \quad \mathbf{1} = \Big(1, 1, \dots, 1\Big)^T, \tag{1.2a}$$

$$\frac{a}{b} = \frac{ac}{bc}, \text{ for any } c \neq 0, \tag{1.2b}$$

$$\text{If } a \neq 0 \quad \text{then} \quad \text{vector } \ a^{-1} = \Big(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\Big)^T, \tag{1.2c}$$

$$\text{well-defined} \ \text{ and } a^{-1}a = \mathbf{1}.$$

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The aim of this study is to construct both matrix- and derivative-free iterations with high efficiency in $\mathbb{R}^n$ using point-wise operations (multiplication and division). That is we aim to construct super-schemes (SS, for short) with $O(n^2)$ complexity, in contrast to most existing iterative methods, which typically have $O(n^3)$ complexity. To our knowledge, such schemes do not exist to date. The remaining parts of this paper are outlined as follows. Section 2 is devoted to the construction of matrix and derivative-free two- and three-step iterations with high-order convergence. In section 3, we consider the first order error analysis for the proposed iterations. In section 4, we propose two- and three-step iterations with scalar coefficients. In section 5, we propose m-step super schemes (for short MSS). In Section 6, we present the results of numerical experiments to demonstrate the performance of SS and to compare them with other well-known iterative methods.

2. Construction of novel two- and three-step iterations

In our previous study [7], we proposed highly efficient derivative-free iterations:

$$\begin{aligned} y_k &= x_k - [x_k, w_k; F]^{-1} F(x_k), \\ x_{k+1} &= y_k - T_k [x_k, w_k; F]^{-1} F(y_k), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} y_k &= x_k - [x_k, w_k; F]^{-1} F(x_k), \\ z_k &= y_k - T_k [x_k, w_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - H_k [x_k, w_k; F]^{-1} F(z_k). \end{aligned} \tag{2.2}$$

where first-order divided difference operator $[x_k, w_k; F]$ is given by

$$\begin{aligned} &[x_k, w_k; F](w_k - x_k) = F(w_k) - F(x_k), \\ &w_k = x_k + \gamma F(x_k),\ \gamma \in \mathbb{R} \setminus \{0\}. \end{aligned} \tag{2.3}$$

It was proven that the local convergence order of (2.1) is four if and only if the parameter T_k satisfies

$$T_k = \mathbf{1} + \Theta_k + r_k + O(h^2), \tag{2.4}$$

where

$$\Theta_k = \frac{F(y_k)}{F(x_k)},\ \ r_k = \frac{F(y_k)}{F(w_k)}, \tag{2.5}$$

which are well defined under condition $F(x_k) \neq 0$. The local convergence order of (2.2) is six if and only if the parameters T_k and H_k satisfy

$$H_k = T_k = \mathbf{1} + \Theta_k + r_k + O(h^2). \tag{2.6}$$

Moreover, the local convergence order of (2.2) is eight if and only if T_k and H_k satisfy condition

$$T_k = \mathbf{1} + \Theta_k + r_k + \beta\Theta_k^2 + \gamma\Theta_k^3 + \ldots,\ \ \beta, \gamma \in \mathbb{R}. \tag{2.7}$$

and

$$H_k = T_k + \Theta_k r_k + (\Theta_k + r_k)((\beta - 1)\Theta_k^2 - r_k^2) + (1 + 2(\Theta_k + r_k))S_k, \tag{2.8}$$

where

$$S_k = \frac{F(z_k)}{F(y_k)},$$

respectively. In $\mathbb{R}^n$ with point-wise operations (multiplication and division), it is possible to consider divided difference $F[x_k, w_k]$ as

$$F[x_k, w_k] = \frac{F(w_k) - F(x_k)}{w_k - x_k}. \tag{2.9}$$

There is a remarkable connection between $F[x_k, w_k]$ and $[x_k, w_k; F]$. Namely, the following result holds.

THEOREM 1. *Let $F(x)$ sufficiently differentiable vector function. Then*

$$F[x_k, w_k] = [x_k, w_k; F](\mathbf{1} + O(h^4)), \quad h = O(F(x_k)). \tag{2.10}$$

Proof. In [7], it was proven that

$$[x_k, w_k; F] = F'(x_k)(I + A_k + B_k + D_k) + O(h^4), \tag{2.11}$$

where

$$\begin{aligned} A_k &= \frac{1}{2}F'(x_k)^{-1}F''(x_k)\gamma F(x_k), \\ B_k &= \frac{1}{6}F'(x_k)^{-1}F'''(x_k)\gamma^2 F^2(x_k), \\ D_k &= \frac{1}{24}F'(x_k)^{-1}F^{IV}(x_k)\gamma^3 F^3(x_k). \end{aligned} \tag{2.12}$$

From (2.11), we get

$$F'(x_k) = [x_k, w_k; F](I + A_k + B_k + D_k)^{-1} + O(h^4), \tag{2.13}$$

or

$$\begin{aligned} \gamma F'(x_k) =& \Delta(I + A_k + B_k + D_k)^{-1} + O(h^4) \\ =& \Delta(I - A_k - B_k - D_k + A_k^2 + 2A_kB_k - A_k^3) + O(h^4), \end{aligned} \tag{2.14}$$

where $\Delta = \gamma[x_k, w_k; F]$.

In what follows, the notation $\dfrac{1}{I + \Delta}$ stand for $(I + \Delta)^{-1}$. Using permutation properties of $q-$ linear derivatives [3], we get

$$A_k = \gamma F'(x_k)\eta_k, \quad B_k = (\gamma F'(x_k))^2 d_k, \quad D_k = \frac{1}{4}(\gamma F'(x_k))^3 c_k, \tag{2.15}$$

where

$$\begin{aligned} \eta_k &= \frac{1}{2}F'(x_k)^{-1}F''(x_k)\xi_k, \quad d_k = -\frac{1}{6}F'(x_k)^{-1}F'''(x_k)\xi_k^2, \\ c_k &= \frac{1}{6}F'(x_k)^{-1}F^{IV}(x_k)\xi_k^3, \quad \xi_k = F'(x_k)^{-1}F(x_k). \end{aligned} \tag{2.16}$$

For η_k, the following expansion holds [7]

$$\begin{aligned} \eta_k = \frac{\Theta_k}{(I + \Delta)} + (\Delta - 1)d_k + \frac{2\Delta}{(1 + \Delta)^2}\Theta_k^2 + \frac{3\Delta^2}{(1 + \Delta)^3}\Theta_k^3 \\ - \Delta d_k\Theta_k - \frac{\Delta^2 - \Delta + 1}{4}c_k + O(h^4), \end{aligned} \tag{2.17}$$

$$A_k = \Delta\eta_k - \Delta^2\eta_k^2 + 2\Delta^3\eta_k^3 + \Delta^3\eta_k d_k + O(h^4), \tag{2.18}$$

$$B_k = -\Delta^2 d_k + 2\Delta^3 \eta_k d_k + O(h^4), \tag{2.19}$$

$$D_k = \frac{\Delta^3}{4} c_k + O(h^4). \tag{2.20}$$

Using the relations (2.17)-(2.20) in (2.14), we get

$$\gamma F'(x_k) = \Delta \left(I - \frac{\Delta}{I+\Delta}\Theta_k + \Delta d_k - \frac{3\Delta^2}{1+\Delta}\Theta_k d_k + \frac{\Delta(1-\Delta)}{4} c_k \right) + O(h^4), \tag{2.21}$$

thereby

$$(\gamma F'(x_k))^2 = \Delta^2 \left(I - \frac{2\Delta}{1+\Delta}\Theta_k + \frac{\Delta^2}{(1+\Delta)^2}\Theta_k^2 + 2\Delta d_k \right) + O(h^3), \tag{2.22}$$

$$(\gamma F'(w_k))^3 = \Delta^3 \left(I - \frac{3\Delta}{1+\Delta}\Theta_k \right) + O(h^2), \tag{2.23}$$

$$(\gamma F'(x_k))^4 = \Delta^4 (I + O(h)). \tag{2.24}$$

On the other hand, it is easy to show that

$$\begin{aligned} \frac{F(w_k)}{F(x_k)} = I + \gamma F'(x_k) + \eta_k (\gamma F'(x_k))^2 \\ - d_k (\gamma F'(x_k))^3 + \frac{1}{4} c_k (\gamma F'(x_k))^4 + O(h^4). \end{aligned} \tag{2.25}$$

If we take (2.21), (2.22), (2.23) and (2.24) into account then from (2.25) we obtain

$$\begin{aligned} \frac{F(w_k)}{F(x_k)} = I + \Delta \left(I - \frac{\Delta}{1+\Delta}\Theta_k + \Delta d_k - \frac{3\Delta^2}{1+\Delta} d_k \Theta_k + \frac{\Delta(1-\Delta)}{4} c_k \right) \\ + \eta_k \Delta^2 \left(I - \frac{2\Delta}{1+\Delta}\Theta_k + \frac{\Delta^2}{(1+\Delta)^2}\Theta_k^2 + 2\Delta d_k \right) \\ - d_k \Delta^3 (I - \frac{3\Delta}{1+\Delta}\Theta_k) + \frac{\Delta^4}{4} c_k + O(h^4). \end{aligned} \tag{2.26}$$

Taylor expansion of $F(w_k)$ at x_k gives

$$\begin{aligned} F(w_k) - F(x_k) = \left(F'(x_k) + \frac{F''(x_k)}{2}\gamma F(x_k) + \frac{F'''(x_k)}{3!}(\gamma F(x_k))^2 \right. \\ \left. + \frac{F^{IV}(x_k)}{4!}(\gamma F(x_k))^3 \right) \gamma F(x_k) + O(h^5). \end{aligned} \tag{2.27}$$

Comparison of (2.3) and (2.27) gives

$$\begin{aligned} [x_k, w_k; F] = F'(x_k) + \frac{F''(x_k)}{2}\gamma F(x_k) + \frac{F'''(x_k)}{3!}(\gamma F(x_k))^2 \\ + \frac{F^{IV}(x_k)}{4!}(\gamma F(x_k))^3 + O(h^4) = F'(x_k)(I \\ + A_k + B_k + D_k) + O(h^4). \end{aligned} \tag{2.28}$$

By definition (see (2.9)) and by virtue of (2.26)

$$\begin{aligned} F[x_k, w_k] =& \frac{1}{\gamma}\left(\frac{F(w_k)}{F(x_k)} - \mathbf{1}\right) = [x_k, w_k; F]\{I - \frac{\Delta}{1+\Delta}\Theta_k \\ &+ \Delta d_k - \frac{3\Delta^2}{1+\Delta} d_k\Theta_k - \frac{\Delta(1-\Delta)}{4} c_k + \\ &+ \eta_k\Delta(I - \frac{2\Delta}{1+\Delta}\Theta_k + \frac{\Delta^2}{(1+\Delta)^2}\Theta_k^2 + 2\Delta d_k) \\ &- \Delta^2 d_k(I - \frac{3\Delta}{1+\Delta}\Theta_k) + \frac{\Delta^3}{4} c_k\} + O(h^4). \end{aligned} \tag{2.29}$$

Substituting η_k given by (2.17) into (2.29), we obtain (2.10). □

We refer to (2.10) as the key transformation, which allows us to simplify the iteration (2.1). Now, we consider two-step iteration:

$$\begin{aligned} y_k &= x_k - \frac{F(x_k)}{F[x_k, w_k]}, \\ x_{k+1} &= y_k - T_k \frac{F(y_k)}{F[x_k, w_k]}. \end{aligned} \tag{2.30}$$

THEOREM 2. *The iteration* (2.1) *is equivalent to* (2.30) *within of accuracy* $O(h^4)$.

Proof. The first substep in (2.1) can be written as

$$y_k - x_k = -[x_k, w_k; F]^{-1} F(x_k).$$

Multiplying both sides of last equation by vector $F[x_k, w_k]$ given by (2.10) and using permutation properties and (1.2a), (1.2b), we get

$$F[x_k, w_k](y_k - x_k) = -[x_k, w_k; F](1 + O(h^4))[x_k, w_k; F]^{-1} F(x_k) = -F(x_k) + 0(h^4),$$

or

$$y_k = x_k - \frac{F(x_k)}{F[x_k, w_k]},$$

that is, by transformation (2.10) the first substep in (2.1) leads to first one in (2.30). The converse is obvious. The equivalence of second sub-step of (2.1) and (2.30) is checked in a similar way, as the first sub-steps. □

To validity of (2.30) we can present the following argument. The two-step iteration (2.30) can be considered as extension of the two-point iteration for scalar equation to the multidimensional case. The fourth-order convergence condition (2.4) leads to

$$T_k = 1 + \frac{f(y_k)}{f(x_k)} + \frac{f(y_k)}{f(w_k)} + O(h^2) = 1 + \hat{d}_k\Theta_k + O(h^2), \;\; \hat{d}_k = \frac{2 + \gamma\phi_k}{1 + \gamma\phi_k},$$

which coincides with fourth-order convergence condition (3.23) in [10] (see also (4.20) in [11]). This fact again demonstrate the validity of scheme (2.30). It should be pointed out that by virtue of (2.29) holds

$$F[x_k, w_k] \neq 0 \;\text{ iff }\; F(x_k) \neq 0,$$

that understood as component-wise and

$$\frac{F(x_k)}{F[x_k,w_k]}=\frac{F(x_k)^2}{F(w_k)-F(x_k)}\simeq\frac{O(h^2)}{O(h)}\xrightarrow{h\to 0}0.$$

Thus, the iteration (2.30) is well-defined under the condition $F(x_k)\neq 0$. Analogously, it is easy to check that, the three-step iteration (2.2) is equivalent to

$$\begin{aligned}y_k&=x_k-\frac{F(x_k)}{F[x_k,w_k]},\\ z_k&=y_k-T_k\frac{F(y_k)}{F[x_k,w_k]},\\ x_{k+1}&=z_k-H_k\frac{F(z_k)}{F[x_k,w_k]},\end{aligned}\tag{2.31}$$

within of accuracy $O(h^8)$. Therefore, the convergence order of iteration (2.31) is equal to six under condition (2.6). The main advantages of schemes (2.30) and (2.31) are that they do not require solving a linear system of equations per iteration and require only multiplication and division of vectors. As a result, the total computation cost is $O(n^2)$, while the total computation cost of schemes (2.1) and (2.2) is $O(n^3)$. They are fully explicit, matrix- and derivative-free, and according to the Definition 1 in [12] they are super schemes (SS). To the best of our knowledge, such a super scheme has not appeared in the literature to date, and we first obtain high efficient SS that considered as reasonably finding in iteration theory. Since the high-order SS are matrix-free and derivative-free, they do not require Jacobian evaluations, significantly reduce memory storage requirements and CPU time. For these reasons, the proposed methods can be successfully applied to high-precision and large-scale problems.

3. First order error analysis of schemes (2.30) and (2.31)

Now, we perform a first order error analysis of scheme (2.30). That is, we formally neglect $O(h^2)$ term. This formal analysis is meaningful if it is sufficiently small. From (2.30) it follows that

$$x_{k+1}=x_k-\frac{F(x_k)}{F[x_k,w_k]}-T_k\frac{\Theta_kF(x_k)}{F[x_k,w_k]}+O(h^2)=x_k-\frac{1}{F[x_k,w_k]}(1+T_k\Theta_k)F(y_k).$$

In the last equation, we can ignore the term with Θ_k because of $\Theta_k=O(h)$. Then, we have

$$x_{k+1}=x_k-\frac{1}{F[x_k,w_k]}F(x_k)+O(h^2).$$

Then, the Taylor expansion of $F(x_{k+1})$ at x_k gives

$$F(x_{k+1})=F(x_k)-F'(x_k)\frac{F(x_k)}{F[x_k,w_k]}=\left(\mathbf{1}-\frac{F'(x_k)\mathbf{1}}{F[x_k,w_k]}\right)F(x_k).\tag{3.1}$$

Using (2.10) , (2.11), (2.17) and (2.18), we get

$$\begin{aligned}\mathbf{1}-\frac{F'(x_k)}{F[x_k,w_k]}&=\mathbf{1}-(I+A_k)^{-1}\approx I-I+A_k=A_k+O(h^2)\\ &=\frac{\Delta\Theta_k}{1+\Delta}=\Theta_k-r_k+O(h^2).\end{aligned}$$

Then from (3.1), we obtain

$$\|F(x_{k+1})\| = \|\Theta_k - r_k\| \cdot \|F(x_k)\|. \tag{3.2}$$

Since Θ_k and r_k are of $O(h)$ then $\Theta_k - r_k$ is high order small as compared to Θ_k and r_k. Therefore $\|\Theta_k - r_k\| < 1$. Thus, from (3.2) we get

$$\|F(x_{k+1})\| < \|F(x_k)\|, \tag{3.3}$$

under condition

$$\|\Theta_k - r_k\| < 1. \tag{3.4}$$

It means that the inequality (3.3) guarantees convergence of sequence $\{x_k\}$ defined by (2.30).

By definition of r_k, from (2.5) the following holds $r_k \to \Theta_k$ when $F(x_k) \to 0$. Thus, one may expect that the inequality (3.4) holds true beginning from some $k \geq k_0$.

As for schemes (2.31), the first two sub-steps are same as (2.30). The third-step of (2.31) increases the convergence order. So, the condition (3.4) also guarantees convergence of scheme (2.31) with sixth-order. The computed vector Θ_k can be used as convergence test. Indeed, from $F(y_k) = \Theta_k F(x_k)$ it clear that

$$\|F(y_k)\| < \|F(x_k)\|, \tag{3.5}$$

if

$$\|\Theta_k\| < 1. \tag{3.6}$$

Similarly, for $F(z_k)$ i.e.,

$$\|F(z_k)\| < \|F(y_k)\|, \quad \text{if} \quad \|S_k\| = \left\| \frac{F(z_k)}{F(y_k)} \right\| < 1.$$

As a consequence, the following chain inequality holds true

$$\|F(z_k)\| < \|F(y_k)\| < \|F(x_k)\|, \tag{3.7}$$

under conditions

$$\|\Theta_k\| < 1 \quad \text{and} \quad \|S_k\| < 1. \tag{3.8}$$

4. Super efficient schemes with scalar coefficients

In order to pass to schemes with scalar coefficients, we first consider one-step iterative method

$$x_{k+1} = x_k - \frac{1}{F[x_k, w_k]} \cdot F(x_k). \tag{4.1}$$

It should be noted that the first sub-step of (2.30) can be rewritten as (4.1). Using property (1.2b)

$$\frac{1}{F[x_k, w_k]} = \frac{F(x_k)(F(w_k) - F(x_k))}{(F(w_k) - F(x_k))(F(w_k) - F(x_k))},$$

and transition rule in [12] one can pass from (4.1) to its variant

$$x_{k+1} = x_k - \alpha_k F(x_k), \tag{4.2}$$

with scalar coefficient α_k given by

$$\alpha_k = \frac{(F(w_k) - F(x_k))^T F(x_k)}{\|F(w_k) - F(x_k)\|^2}. \tag{4.3}$$

Another suitable choice is

$$\alpha_k = \frac{\|F(x_k)\|^2}{F(x_k)^T (F(w_k) - F(x_k))}, \tag{4.4}$$

that can be used in (4.2) instead of (4.3). To study the convergence properties of iteration (4.2), we take

$$\phi(\alpha) = \|F(x_k - \alpha F(x_k))\|^2, \tag{4.5}$$

as merit function and aim to find the parameter α by minimizing (4.5). The Taylor expansion of $F(x)$ at point x_k gives

$$\begin{aligned} F(x_k - \alpha F(x_k)) &= F(x_k) - \alpha F'(x_k) F(x_k) + O(h^2) \\ &= F(x_k) - \alpha (F(w_k) - F(x_k)) + O(h^2), \end{aligned} \tag{4.6}$$

in which we used secant approximation

$$\frac{F(w_k) - F(x_k)}{F(x_k)} = F'(x_k)(\mathbf{1} + O(h)),$$

Substituting (4.6) into (4.5) and neglecting small term of $O(h^2)$, we obtain

$$\phi(\alpha) = \|F(x_k) - \alpha (F(w_k) - F(x_k))\|^2, \tag{4.7}$$

instead of (4.5). The function (4.7) has a minimum at α defined by (4.3) and the minimum value $\phi(\alpha_k)$ is

$$\|F(x_k - \alpha_k F(x_k)\|^2 \approx \|F(x_k)\|^2 - \frac{(F(w_k) - F(x_k))^T F(x_k)^2}{\|F(w_k) - F(x_k)\|^2} < \|F(x_k)\|^2. \tag{4.8}$$

i.e., the parameter α_k given by (4.3) approximately minimizes the residual norm $\|F(x_k + \alpha F(x_k))\|^2$. We call α_k the optimal step-size of (4.2) and it reduces the risk of divergence and enlarges the convergence domain of iteration (4.2). When the initial guess is far from the exact root x^*, utilizing optimal step-sizes is often necessary to attain global convergence.

Remark 1. *The iteration* (4.2) *can be considered as steepest descent method with an optimal line search, given by* (4.3).

Based on the above considered argument, we consider two-and three-step iterations equipped with an optimal step size.

$$\begin{aligned} y_k &= x_k - \alpha_k F(x_k), \\ x_{k+1} &= y_k - \alpha_k T_k F(y_k), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} y_k &= x_k - \alpha_k F(x_k), \\ z_k &= y_k - \alpha_k T_k F(y_k), \\ x_{k+1} &= z_k - \alpha_k H_k F(z_k). \end{aligned} \tag{4.10}$$

where α_k is defined by (4.3) and T_k satisfies (2.6). Of course, the local order convergence of (4.9) and (4.10) is the same as of (2.30) and (2.31). That is, $\rho = 4$ and $\rho = 6$ for iterations (4.9) and (4.10), respectively, under choices (4.3) and (2.6). Moreover, $\rho = 8$ for iteration (4.10) under the parameter choices (2.7), (2.8) and (4.3).

As mentioned above, the optimal step-size α_k ensures global convergence of iteration, while parameter T_k accelerates convergence speed of iterations (4.9) and (4.10) from two to six.

It is possible to pass from (4.9) and (4.10) to their variants with scalar coefficients and here we present the final form

$$\begin{aligned} y_k &= x_k - \alpha_k F(x_k), \\ x_{k+1} &= y_k - \alpha_k \beta_k F(x_k), \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} y_k &= x_k - \alpha_k F(x_k), \\ z_k &= y_k - \alpha_k \beta_k F(x_k), \\ x_{k+1} &= z_k - \alpha_k \gamma_k (\delta_k + \beta_k) F(x_k), \end{aligned} \tag{4.12}$$

where

$$\beta_k = \frac{\|F(y_k)\|^2}{\|F(x_k)\|^2} + \frac{\|F(y_k)\|^2}{F(x_k)^T F(w_k)},\ \gamma_k = \frac{\|F(z_k)\|^2}{F(z_k)^T F(y_k)},\ \delta_k = \frac{\|F(y_k)\|^2}{F(y_k)^T F(x_k)}. \tag{4.13}$$

5. Multi-step super schemes (MSS)

The previously constructed two- and three-step super schemes (2.30) and (2.31) allows us to define m-step ones. We now consider the following iteration

$$\begin{aligned} \psi_1^k &= x_k - \alpha_k F(x_k) \\ \psi_i^k &= \psi_{i-1}^k - \alpha_k T_k F(\psi_{i-1}^k),\ \ i = 2, 3, ..., m, \\ x_{k+1} &= \psi_m^k, \end{aligned} \tag{5.1}$$

with T_k satisfying condition (2.6) i.e.,

$$T_k = \mathbf{1} + \Theta_k + r_k + O(h^2). \tag{5.2}$$

THEOREM 3. *Let the parameters α_k and T_k in* (5.1) *are given by* (4.3) *and* (5.2) *Then the order of local convergence of iteration* (5.1) *is equal to* $2m$.

Proof. From (2.31) it clear that; $F(\psi_2^k) = O(h^4)$ and $F(\psi_3^k) = O(h^6)$ i.e., the convergence order are $\rho = 4$ and 6 for $i = 2, 3$. Let $F(\psi_j^k) = O(h^{2j})$. Then according to choice (2.6) the convergence order of next iteration step increased by two units. That is $F(\psi_{j+1}^k) = O(h^{2j+2}) = O(h^{2(j+1)})$. Repeated application of this result gives $F(x_{k+1}) = F(\psi_m^k) = O(h^{2m})$. □

If, instead of T_k in (5.1) we take

$$T_k = \mathbf{1} + \Theta_k + r_k + \Theta_k^2 + 4\Theta_k r_k + \sigma r_k^2 + t_k \frac{\mathbf{1} + r_k - \frac{1}{2}(\Theta_k + e_k)}{2 \cdot \mathbf{1} - t_k} + O(h^3), \tag{5.3}$$

where

$$e_k = \frac{F(\Omega_k)}{F(x_k)}, \;\; \Omega_k = x_k + \frac{F(x_k)}{F[x_k, w_k]} = 2x_k - y_k, \;\; t_k = \frac{F(w_k)}{F(x_k)} \tag{5.4}$$

and

$$\sigma = \begin{cases} 0 & \text{when } \; i = 2 \\ 1 & \text{otherwise} \end{cases}, \tag{5.5}$$

then the convergence order of (5.1) equals $3m - 1$. Indeed, the choice T_k for $i = 2$ guarantees $F(\psi_2^k) = O(h^5)$ and for $i \geq 3$ guarantees $F(\psi_3^k) = O(h^8)$, $F(\psi_4^k) = O(h^{11}), \ldots,$ and $F(\psi_m^k) = F(x_{k+1}) = O(h^{3m-1})$.

6. Numerical experiments and efficiency index

We illustrate the efficiency of the proposed SS and confirm the associated theoretical results. Furthermore, the methods of orders 4, 6, 7, and 8 in the experiment are denoted by SS4, SS6, SS7, and SS8, respectively (T_k and H_k are given in (2.7) and (2.8); see Table 5 in [7]). In addition, method (5.1) with $m = 5$ is denoted by MSS10, while the scalar variants of SS given by methods (4.9) and (4.10) are denoted by SCSS4 and SCSS6, respectively.

All of our numerical computations were performed in programming system MATHEMATICA 14.1 using multi-precision arithmetic with 1000 digits for Intel(R) i5-13700, 2.10GHz, 32GB RAM. In all experiments, the iterative process was terminated when the following stopping criterion was satisfied:

$$\|x_{k-1} - x_k\| \leq 10^{-60}.$$

The computational order of convergence (ACOC) at iteration k is defined by:

$$\rho_k = \frac{\log\left(\|F(x_{k+1})\|/\|F(x_k)\|\right)}{\log\left(\|F(x_k)\|/\|F(x_{k-1})\|\right)}.$$

For the numerical experiments, we consider the following test problem:

EXAMPLE 1. *Consider the nonlinear system:*

$$\begin{aligned} &x_i^2 x_{i+1} - 1 = 0, \; i = 1, 2, \ldots, n-1, \\ &x_n^2 x_1 - 1 = 0. \end{aligned}$$

The exact solution is $(1, 1, \ldots, 1)^T$ *and the initial approximation is* $x_0 = (1.25, 1.25, \cdots, 1.25)^T$.

EXAMPLE 2. *Consider the system of equations:*

$$\begin{cases} x_i + \log(2 + x_i + x_{i+1}) = 0, \quad 1 \leq i \leq n-1, \\ x_n + \log(2 + x_n + x_1) = 0. \end{cases}$$

Starting value is $x_0 = (0.1, 0.1, 0.1, 0.1, \ldots, 0.1)^T$.

EXAMPLE 3. *We consider the following system of nonlinear equations involving trigonometric functions:*

$$\begin{cases} x_i \sin(x_{i+1}) - 1 = 0,\ i = 1, 2, \ldots, n-1 \\ x_n \sin(x_1) - 1 = 0 \end{cases}$$

The exact solution is $(1.11415714, 1.11415714, \ldots, 1.11415714)^T$ *and* $x_0 = (1.3, 1.3, \cdots, 1.3)^T$ *is the initial approximation.*

EXAMPLE 4. *We consider the singular boundary value problem (SBVP), known as the Lane-Emden type equation:*

$$\begin{cases} y''(x) + \dfrac{2}{x} y'(x) - 6y(x) = 4y(x)\ln(y(x)), & x \geq 0, \\ y(0) = 1,\ y'(0) = 0. \end{cases}$$

After applying finite difference schemes, the problem is reduced to a system of $n-1$ *nonlinear equations with* $n-1$ *unknowns:*

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \frac{2}{x_i}\,\frac{y_{i+1} - y_{i-1}}{2h} - 6y_i = 4y_i \ln(y_i),\ i = 1, 2, \ldots, n-1.$$

We set $n = 3000$ *and take the initial guess* $(1, 1, \cdots, 1)^T$.

Table 1: Numerical Results for Example 1 (Problem Size n=100000)

Method	CPU Time	k	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
SS4	25.64	5	1.61×10^{-20}	7.21×10^{-85}	4.00
SCSS4	23.766	4	6.47×10^{-20}	7.21×10^{-85}	4.00
SS6	29.141	4	3.35×10^{-17}	4.75×10^{-108}	6.00
SCSS6	29.297	5	6.34×10^{-108}	5.29×10^{-625}	6.00
SS7	29.875	4	8.48×10^{-50}	2.01×10^{-354}	7.00
SS8	34.797	4	1.04×10^{-41}	1.74×10^{-340}	8.00
MSS10	96.210	4	4.15×10^{-87}	5.12×10^{-871}	10.00

Table 2: Numerical Results for Example 2 (Problem Size n=50000)

Method	CPU Time	k	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
SS4	23.203	5	1.39×10^{-21}	3.66×10^{-90}	4.00
SS6	31.094	4	1.31×10^{-73}	3.53×10^{-448}	6.00
SS7	31.36	4	3.73×10^{-116}	5.71×10^{-821}	7.00
SS8	25.125	3	5.12×10^{-21}	1.65×10^{-176}	8.00

Our numerical results are fully consistent with the theoretical analysis and the convergence conditions presented in the preceding sections. From Tables 1–3, we observe that the SS perform remarkably well on problems with dimensions exceeding 5000, whereas most other existing higher-order methods perform poorly on such large-scale problems. In particular, these tables show that SS7 and SS8 are not only faster but also

Table 3: Numerical Results for Example 3 (Problem Size n=10000)

Method	CPU Time	k	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
SS4	7.609	4	7.60×10^{-46}	4.11×10^{-190}	4.00
SS6	9.828	4	2.85×10^{-140}	6.79×10^{-849}	6.00
SS7	7.609	3	4.72×10^{-30}	1.88×10^{-219}	7.00
SS8	7.922	3	5.66×10^{-37}	1.29×10^{-305}	8.00

Table 4: Numerical Results for Example 4 (Problem Size n=3000)

Method	CPU Time	k	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
SS4	7.938	6	1.81×10^{-60}	8.911×10^{-243}	4.00
SS6	8.781	5	9.30×10^{-100}	3.56×10^{-600}	6.00
SS7	7.047	4	1.71×10^{-22}	1.88×10^{-219}	7.00
SS8	7.359	4	7.60×10^{-37}	4.37×10^{-297}	8.00

Table 5: Numerical Results for Example 3 (Problem Size n=200)

Method	Initial vector	k	$\|F(x_k)\|$	ACOC
SS4	x_0	4	5.82×10^{-191}	4.00
SS6	x_0	4	9.60×10^{-650}	6.00
SS7	x_0	3	2.66×10^{-778}	7.00
SS8	x_0	3	8.94×10^{-901}	8.00
SS4	$10x_0$	8	1.32×10^{-179}	4.00
SS6	$10x_0$	7	3.70×10^{-1008}	6.00
SS7	$10x_0$	7	2.16×10^{-1019}	7.00
SS8	$10x_0$	6	1.14×10^{-351}	8.00

Table 6: Example 1 (Problem Size n=10000)

Iteration	SS6	SCSS6	Residual norm
k	$\|\Theta_k - r_k\|$	$\|\Theta_k\|$	$\|F(x_k)\|$
0	47.8028	53.0157	8.18424
1	7.50788	9.70299	0.000233030
2	0.000233030	0.000310706	2.31×10^{-31}
3	2.30×10^{-31}	3.07×10^{-31}	2.18×10^{-193}

require fewer iterations and achieve higher accuracy. Table 4 shows that the SS methods perform well not only on academic test problems but also on the singular boundary value problem (SBVP). Additionally, Tables 5, 6 show that the SS methods still converge when the initial approximation is ten times farther away. Finally, Table 1 also shows that the scalar variants of SS, namely SCSS4 and SCSS6, perform well on large-scale problems while Table 6 confirms that they satisfy the convergence conditions of SS presented in the preceding sections.

In order to compare the performance of different methods, we also consider the computational efficiency index [15] defined as

$$CI = \rho^{\frac{1}{C}},$$

where ρ denotes the order of convergence and C represents the computational cost of each method. Figure 1 illustrates the efficiency index CI for systems of varying sizes. We will investigate the computational efficiency of SS8 and compare it with the methods in [1], [9]

and [14]. In Figure 1, the efficiency indices of the SS8, BS8, EK4S, and ST8S methods are represented by red, green, magenta, and blue colors, respectively. From Figure 1, the SS8 method exhibits remarkably higher computational efficiency and superior overall performance compared with well-known BS8, EK4S, and ST8S methods in [1], [9] and [14]. This advantage arises from its $O(n^2)$ computational complexity, whereas most other existing iterative methods require $O(n^3)$ operations.

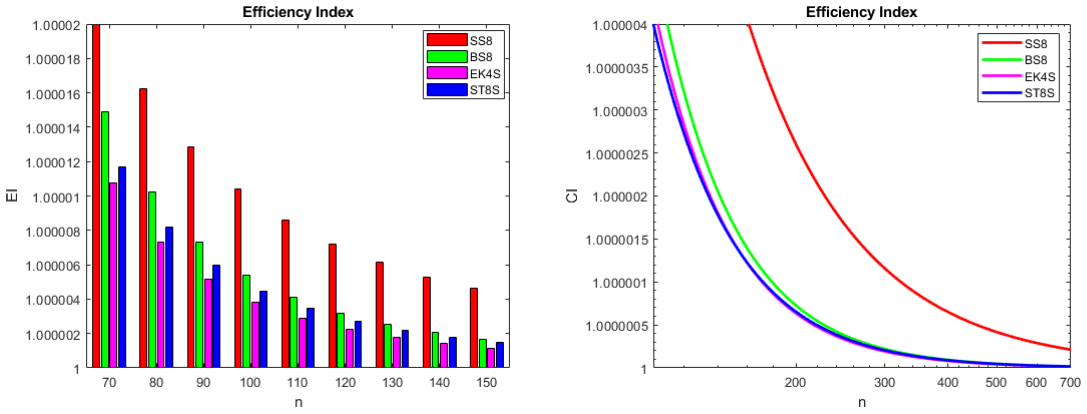

Figure 1: Efficiency indices for different sizes of the system.

7. Conclusions

In this paper, we propose higher-order, super-efficient iterative methods for solving systems of nonlinear equations. In our view, this represents a significant finding in iteration theory and numerical analysis. They do not involve expensive operations such as matrix inversion or multiplication and require only vector operations. They are easy to implement with $O(n^2)$ complexity, whereas all other existing iterative methods have $O(n^3)$ complexity. The numerical experiments are given to demonstrate the effectiveness and performance of proposed iterations.

The many advantages of the proposed SS are as follows:

- They are matrix-free and avoid expensive matrix operations such as inversion and multiplication. No Jacobian evaluation is required and the memory demand is substantially reduced compared with matrix-based methods. While other existing methods require jacobian evaluation and non-singularity assumptions
- Due to fully explicit property, they can be easily implemented using vector operations with low computational cost.
- Higher order convergence and $O(n^2)$ complexity of SS make it particularly useful in high-precision and high-dimensional problems.
- Numerical experiments demonstrated that the SS outperforms other methods in terms of accuracy, CPU time and complexity.

– The SS scheme can be considered a representative of a new generation of iterative methods for solving nonlinear systems, and it will undoubtedly be used to solve many problems in science and engineering.

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