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Design and extension of higher-Order derivative-free iterative methods for nonlinear systems

Tugal Zhanlav, Khuder Otgondorj*

Abstract. In this paper, we propose derivative-free two- and three-step iterative methods with easy implementation. Moreover, we suggest suitable parameter choices that guarantee a local convergence order ρ from four to eight. They require only one matrix inversion at each iteration step and belong to the class of best iterations with high efficiency indices. Several of the proposed algorithms are multidimensional extensions of well-known scalar iterative methods. Numerical experiments are presented to verify the theoretical orders of convergence and to demonstrate the computational efficiency of the proposed methods.

1. Introduction

In the last two decades, many derivative-free and derivative-based iterative methods for solving systems of nonlinear equations have been developed [1–3, 10–21], and references therein. In particular, since 2020, iterative methods in $\mathbb{R}^n$ with point-wise operations (multiplication and division) have been proposed [4, 5, 9–12]. Main advantages of these iterations are that they belong to the class of high efficient methods due to their vector coefficients. Motivated by this, in this paper we aim to develop derivative-free iterations with a simple and useful implementation algorithm, which is very important from the view of practical applications. The local convergence analysis is based on some assumptions about the operator F and utilizes properties of the divided difference operator and permutation properties of the $q-$ derivative $(q \geq 1)$ of F [3]. The structure of the paper is organized as follows. Section 2 is devoted to the construction of two-step derivative-free iterations. We formulate the necessary and sufficient conditions for two-step methods to be fourth- and fifth-order of convergence. The proposed fourth order iterations turn out to be extensions of two-point iterations for solving nonlinear equations to multidimensional cases. In section 3 the higher-order derivative-free three-step iterations are considered. We propose a family of derivative-free iterations with ρ order of convergence ($\rho = 6, 7, 8$) and with high efficiency. The obtained iterations are extensions of some well-known iterations for solving nonlinear equations to multidimensional cases. In section 4 we propose a family of concrete sixth and seventh- and eighth-order derivative-free iterations with an easy implementation algorithm. In section 5 we present results of numerical experiments to confirm the convergence order and performance of proposed methods.

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2. Two-step derivative-free iterations

We consider two-step iteration

$$\begin{aligned} y_k &= x_k - [x_k, w_k; F]^{-1} F(x_k), \\ x_{k+1} &= y_k - T_k [x_k, w_k; F]^{-1} F(y_k), \end{aligned} \tag{2.1}$$

where $[x_k, w_k; F]$ is the first-order divided difference operator, i.e.,

$$\begin{aligned} &[x_k, w_k; F](w_k - x_k) = F(w_k) - F(x_k), \\ &w_k = x_k + \gamma F(x_k), \ \gamma \in \mathbb{R} \setminus \{0\}, \end{aligned} \tag{2.2}$$

and T_k is free parameter (or coefficient of iteration (2.1)). In the following we assume that $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently differentiable and D is an open and convex subset of $\mathbb{R}^n$. $F'(x)$ is nonsingular at point x^* and $x_0 \in \mathbb{R}^n$ is close enough to solution x^* of the equation $F(x) = 0$. For local convergence analysis often used Taylor expansion of vector function F and permutation properties of q-derivative of F. Let $\mathbb{R}^n$ be n-dimensional space with multiplication and division operations of vectors [4, 5, 9–12].

In [7] it was shown that the two-step derivative-free iteration converges with fourth and fifth-order if only if the parameter T_k satisfies

$$T_k = I + 2\eta_k + A_k + O(h^2), \tag{2.3}$$

and

$$T_k = I + 2\eta_k + 5\eta_k^2 + 3d_k + A_k + B_k + 2\eta_k A_k + O(h^3), \ h = O(F(x_k)), \tag{2.4}$$

respectively. Here

$$\begin{aligned} \eta_k &= \frac{1}{2} F'(x_k) F''(x_k) F'(x_k)^{-1} F(x_k), \\ d_k &= -\frac{1}{6} F'(x_k)^{-1} F'''(x_k) (F'(x_k)^{-1} F(x_k))^2, \end{aligned} \tag{2.5}$$

$$\begin{aligned} A_k &= \frac{1}{2} F'(x_k)^{-1} F''(x_k) \gamma F(x_k), \\ B_k &= \frac{1}{6} F'(x_k)^{-1} F'''(x_k) \gamma^2 F(x_k)^2. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we see that $\eta_k = O(h), \ A_k = O(h), \ d_k = O(h^2), \ B_k = O(h^2)$. Our task is to express T_k by means of vectors:

$$\Theta_k = \frac{F(y_k)}{F(x_k)}, \ \ r_k = \frac{F(y_k)}{F(w_k)}, \ \ t_k = \frac{F(w_k)}{F(x_k)}, \tag{2.7}$$

and prove the following.

Theorem 1. *The local convergence order of iteration* (2.1) *is equal to four, if and only if* T_k *satisfies*

$$T_k = \mathbf{1} + 2l_k + O(h^2), \tag{2.8}$$

where

$$l_k = \frac{\Theta_k + r_k}{2}.$$

Proof. Taylor expansion of $F(w_k)$ at point x_k and first order divided difference given by (2.2) gives

$$[x_k, w_k; F] = F'(x_k)(I + A_k) + O(h^2), \quad A_k = O(h). \tag{2.9}$$

Then, we obtain

$$[x_k, w_k; F]^{-1}F(x_k) = (I - A_k)\xi_k + O(h^2), \quad \xi_k = F'(x_k)^{-1}F(x_k). \tag{2.10}$$

By virtue of permutation properties of $q-$derivative, we have

$$A_k = \gamma F'(x_k)\eta_k = \gamma[x_k, w_k; F](I - A_k)\eta_k = \Delta\eta_k + O(h^2). \tag{2.11}$$

By using Taylor expansion of $F(y_k)$ at point x_k and (2.11) one can easily obtain

$$\Theta_k = A_k + \eta_k + O(h^2) = (1 + \Delta)\eta_k + O(h^2),$$

or

$$\eta_k = \frac{\Theta_k}{1 + \Delta} + O(h^2). \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.3), we obtain

$$T_k = \mathbf{1} + \frac{2 + \Delta}{1 + \Delta}\Theta_k + O(h^2), \quad \Delta = \gamma[x_k, w_k; F]. \tag{2.13}$$

$$\frac{2 + \Delta}{1 + \Delta}\Theta_k = \Theta_k + r_k \tag{2.14}$$

Using (2.14) in (2.13), we obtain (2.8). □

In what following the notation $\dfrac{1}{I + \Delta}$ understood as $(I + \Delta)^{-1}$. The iteration (2.1) with parameter given by (2.8) can be considered as extension of schemes given in [8] to multidimensional case.

Remark 1. *If $T_k = \mathbf{1} + O(h)$, then the order of the iteration* (2.1) *decreases by one unit; that is, it has third-order convergence.*

Remark 2. *In some cases, γ is represented in a variable form as: $\gamma = \gamma_k = (F(x_k))^m$, $m \geq 1$ see, for example [16]. For $m = 1$, we have $\Delta = \gamma[x_k, w_k; F] = F(x_k)[x_k, w_k; F] = O(h)$. Therefore, the formula* (2.8) *becomes*

$$T_k = \mathbf{1} + 2\Theta_k + O(h^2). \tag{2.15}$$

Remark 3. *Instead of* (2.8) *one can use the following parameterized expression*

$$T_k = \frac{\mathbf{1} + al_k + bl_k^2}{\mathbf{1} + (a - 2)l_k + dl_k^2}, \quad a, b, d \in \mathbb{R}. \tag{2.16}$$

Thus, we have a family of fourth-order iterations (2.1) with (2.16), which contain four parameter γ, a, b and d. When $\Delta = 0$, $l_k = \Theta_k$ the formula (2.16) leads to

$$T_k = \frac{\mathbf{1} + a\Theta_k + b\Theta_k^2}{\mathbf{1} + (a - 2)\Theta_k + d\Theta_k^2}, \quad a, b, d \in \mathbb{R}, \tag{2.17}$$

which is the well-known necessary and sufficient condition [10], [11] for fourth order two-step iteration

$$y_k = x_k - F'(x_k)^{-1}F(x_k)$$
$$x_{k+1} = y_k - T_k F'(x_k)^{-1}F(y_k). \quad (2.18)$$

Now, we consider some particular cases of (2.16).

(1) $b = d = 0$ and $a = \beta, \ \beta \in \mathbb{R}$. Then (2.16) becomes as:

$$T_k = \frac{\mathbf{1} + \beta l_k}{\mathbf{1} - (\beta - 2)l_k}. \quad (2.19)$$

The iteration (2.1) with parameter T_k given by (2.19) is derivative-free extension of King's iterations.

(2) Let $b = 0, \ d = 1, \ a = 0$. Then (2.16) becomes as:

$$T_k = \frac{\mathbf{1}}{(\mathbf{1} - l_k)^2}. \quad (2.20)$$

The iteration (2.1) with parameter T_k given by (2.20) is derivative-free extension of Kung and Tranb's iterations.

(3) Let $a = 1, \ b = -1, \ d = 0$. Then (2.16) becomes as:

$$T_k = \frac{\mathbf{1} + l_k - l_k^2}{\mathbf{1} - l_k}. \quad (2.21)$$

The iteration (2.1) with parameter given by (2.21) is derivative-free extension of Maheshwari's iteration.

Besides (2.19), (2.20), and (2.21), we propose concrete schemes (2.1) with T_k satisfying the condition (2.8):

$$T_k = 1 + \frac{\Theta_k + r_k}{1 + c\Theta_k + dr_k}, \quad (2.22a)$$

$$T_k = \frac{[x_k, w_k; F]F[x_k, w_k]}{F[x_k, y_k]F[y_k, w_k]}, \quad (2.22b)$$

$$T_k = \frac{1}{[x_k, w_k; F]^{-1}(F[y_k, w_k] + F[x_k, y_k] - F[x_k, w_k])}, \quad (2.22c)$$

$$T_k = \frac{1 - 2\Theta_k}{1 - 3\Theta_k} \frac{F[x_k, w_k]}{[x_k, w_k; F]^{-1}F[y_k, w_k]}, \quad (2.22d)$$

where $F[u_k; v_k] = \dfrac{F(v_k) - F(u_k)}{v_k - u_k}$.

Table 1: for T_k

T_k	β
(2.22a)	$-[c + \frac{c+d}{1+\Delta} + \frac{d}{(1+\Delta)^2}]$
(2.22b)	$\frac{\Delta^2 + 3\Delta + 3}{(1+\Delta)^2}$
(2.22c)	$(\frac{2+\Delta}{1+\Delta})^2$
(2.22d)	$3 + \frac{1}{1+\Delta} + \frac{1}{(1+\Delta)^2}$

where

$$\begin{aligned}
&\frac{F(w_k)}{F(x_k)} = (I+\Delta)(\mathbf{1}+O(h^3)),\\
&F[x_k,y_k] = [x_k,w_k;F](\mathbf{1}-t_1),\\
&F[x_k,w_k] = [x_k,w_k;F](\mathbf{1}+O(h^3)),\\
&F[y_k,w_k] = [x_k,w_k;F]\big(1-\frac{1}{1+\Delta}\big)+O(h^3),\\
&\frac{F(y_k)}{F(w_k)} = \frac{\Theta_k}{1+\Delta}+O(h^3).
\end{aligned} \tag{2.23}$$

The iteration (2.1) with T_k given by (2.22a) ($c = d = 0$) can be considered as extension of (K-A) Khattri-Agarwal iteration in [14]. The iteration (2.1) with T_k given by (2.22b) is extension of derivative-free iteration given in [15]. To obtain fifth-order iteration (2.1), we aim to express (2.4) by means of vectors

$$\Theta_k = \frac{F(y_k)}{F(x_k)},\ \ \frac{F(\Omega_k)}{F(x_k)},\ \ \Omega_k = 2x_k - y_k \ \text{ and } \ \frac{F(w_k)}{F(x_k)}. \tag{2.24}$$

If we take next $O(h^2)$ term in (2.9), (2.10) and (2.11) into account, we get

$$[x_k,w_k;F]^{-1}F(x_k) = (I - A_k - B_k + A_k^2)F'(x_k)^{-1}F(x_k) + O(h^4). \tag{2.25}$$

As a consequence, we have

$$A_k = \frac{\Delta\eta_k}{1+\Delta\eta_k} + O(h^3) = \Delta\eta_k - \Delta^2\eta_k^3 + O(h^3),\ \ B_k = -\Delta^2 d_k + O(h^3). \tag{2.26}$$

Taylor expansion of $F(y_k)$, $F(\Omega_k)$ and $F(w_k)$ at point x_k give

$$\Theta_k = \frac{F(y_k)}{F(x_k)} = R_k + \eta_k(I - 2A_k) + d_k + O(h^3), \tag{2.27}$$

$$\frac{F(\Omega_k)}{F(x_k)} = 2I - R_k + \eta_k(I - 2A_k) - d_k + O(h^3), \tag{2.28}$$

$$\frac{F(w_k)}{F(x_k)} = (I+\Delta)(I - R_k) + \Delta^2\eta_k(I - 2A_k) - \Delta^3 d_k + O(h^3), \tag{2.29}$$

where

$$R_k = A_k + B_k - A_k^2. \tag{2.30}$$

By virtue of (2.26), we have

$$R_k = \Delta\eta_k - \Delta^2 d_k - 2\Delta^2\eta_k^2 + O(h^3), \tag{2.31}$$

Using (2.31) in (2.27), we get

$$\eta_k = \frac{1}{1+\Delta}\Theta_k + (\Delta - 1)d_k + 2\Delta\eta_k^2 + O(h^3). \tag{2.32}$$

Thereby

$$\eta_k^2 = \frac{1}{(1+\Delta)^2}\Theta_k^2 + O(h^3). \tag{2.33}$$

Also, from (2.27)-(2.29), we find

$$\begin{aligned} d_k &= \frac{2\cdot\mathbf{1} + 2\dfrac{F(y_k)}{F(w_k)} - \Theta_k - \dfrac{F(\Omega_k)}{F(x_k)}}{2\Big(2\cdot\mathbf{1} - \dfrac{F(w_k)}{F(x_k)}\Big)} + O(h^3) \\ &= \frac{\mathbf{1} + r_k - \frac{1}{2}\Big(\Theta_k + \dfrac{F(\Omega_k)}{F(x_k)}\Big)}{2\cdot\mathbf{1} - t_k} + O(h^3). \end{aligned} \tag{2.34}$$

We are now ready to prove the following result.

THEOREM 2. *The local convergence order of iteration* (2.1) *equal to five, if and only if the parameter* T_k *satisfies*

$$T_k = \mathbf{1} + \Theta_k + r_k + \Theta_k^2 + 4\Theta_k r_k + t_k d_k + O(h^3), \tag{2.35}$$

where d_k *is given by formula* (2.34).

Proof. To prove theorem it is enough to show that the condition (2.4) is equivalent to (2.35). Substituting (2.33) into (2.32), we have

$$\eta_k = \frac{1}{1+\Delta}\Theta_k + (\Delta-1)d_k + \frac{2\Delta}{(1+\Delta)^2}\Theta_k^2 + O(h^3). \tag{2.36}$$

By (2.31), we get

$$\eta_k A_k = \Delta\eta_k^2 + O(h^3), \tag{2.37}$$

$$A_k + B_k = \Delta\eta_k - \Delta^2\eta_k^2 - \Delta^2 d_k + O(h^3). \tag{2.38}$$

Substituting (2.37) and (2.38) into (2.4), we get

$$T_k = \mathbf{1} + (2+\Delta)\eta_k + (5 + 2\Delta - \Delta^2)\eta_k^2 + (3-\Delta^2)d_k + O(h^3).$$

Using (2.33) and (2.36) on last expression, we arrive at

$$T_k = \mathbf{1} + \frac{2+\Delta}{1+\Delta}\Theta_k + \frac{5+6\Delta+\Delta^2}{(1+\Delta)^2}\Theta_k^2 + (1+\Delta)d_k + O(h^3). \tag{2.39}$$

Using (2.14) in (2.39), we get (2.35). The converse is obvious. □

When $\Delta = 0$ or $\gamma = 0$ the iteration (2.1) leads to

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - T_k F'(x_k)^{-1}F(y_k), \end{aligned} \tag{2.40}$$

and the condition (2.35) becomes as

$$T_k = \mathbf{1} + 2\Theta_k + 5\Theta_k^2 + d_k + O(h^3). \tag{2.41}$$

In passing, we obtain the following result.

THEOREM 3. *The local convergence order of the iteration* (2.40) *equal to five, if and only if the parameter* T_k *satisfies* (2.41)*, in which* d_k *is defined by*

$$d_k = \mathbf{1} - \frac{1}{2}\left(\frac{F(\Omega_k)}{F(x_k)} - \Theta_k\right) + O(h^3). \tag{2.42}$$

This is a direct consequence of Theorem 2.

3. Three step derivative-free iterations

Now, we consider three-step iterations

$$\begin{aligned} y_k &= x_k - [x_k, w_k; F]^{-1} F(x_k), \\ z_k &= y_k - T_k [x_k, w_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - H_k [x_k, w_k; F]^{-1} F(z_k). \end{aligned} \tag{3.1}$$

The first two sub steps in (3.1) are the same as in (2.1). T_k and H_k are free parameters. Assume that $F(z_k) = O(h^p)$ $p \geq 3$. Then it is known that [6] the local convergence order of iterations (3.1) equal to $p+2$, $p+3$ and $p+4$ if and only if H_k satisfies

$$H_k = I + 2\eta_k + A_k + O(h^2), \tag{3.2a}$$

$$H_k = I + 2\eta_k + 6\eta_k^2 + 3d_k + A_k + B_k + 2\eta_k A_k + O(h^3), \tag{3.2b}$$

$$\begin{aligned} H_k = I &+ 2\eta_k + 6\eta_k^2 + 3d_k + C_k + 20\eta_k^3 + 20\eta_k d_k + A_k + B_k, \\ &+ D_k + 2\eta_k(A_k + B_k) + (6\eta_k^2 + 3d_k)A_k + O(h^4), \end{aligned} \tag{3.2c}$$

where η_k, d_k, A_k and B_k are given by (2.5), (2.6) and

$$\begin{aligned} C_k &= \frac{1}{6} F'(x_k) F^{(IV)}(x_k) (F'(x_k)^{-1} F(x_k))^3, \\ D_k &= \frac{1}{24} F'(x_k) F^{(IV)}(x_k) (\gamma F(x_k))^3. \end{aligned} \tag{3.3}$$

A direct consequence of Theorem 1, 2 and Theorem 3 in [7] is the following.

THEOREM 4. *The local convergence order of iteration* (3.1) *equal to six if and only if* T_k *and* H_k *satisfy the following condition*

$$T_k = H_k = \mathbf{1} + 2l_k + O(h^2). \tag{3.4}$$

REMARK 4. *Theorem 4 is valid for the following choices*

$$\begin{aligned} T_k &= \frac{\mathbf{1} + a_1 l_k + b_1 l_k^2}{\mathbf{1} + (a_1 - 2) l_k + d_1 l_k^2}, \\ H_k &= \frac{\mathbf{1} + \overline{a}_2 l_k + b_2 l_k^2}{\mathbf{1} + (\overline{a}_2 - 2) l_k + d_2 l_k^2}; \quad a_1, b_1, c_1, \overline{a}_2, b_2, d_2 \in R \end{aligned} \tag{3.5}$$

THEOREM 5. *The iterative method* (3.1) *has local convergence order seven (respectively, eight) if and only if* T_k *satisfies* (2.35) *and* H_k *satisfies*

$$H_k = \mathbf{1} + \Theta_k + r_k + O(h^2), \tag{3.6}$$

and

$$H_k = \mathbf{1} + \Theta_k + r_k + \Theta_k^2 + 4\Theta_k r_k + r_k^2 + t_k d_k + O(h^3), \tag{3.7}$$

where d_k *is defined in* (2.42).

Proof. The seventh-order of convergence is obvious, since $F(z_k) = O(h^5)$ i.e., $p = 5$ according to Theorem 2 and $(p+2)$ convergence condition (3.2a) can be rewritten as (3.6) in the same way as (2.8) for T_k. Analogously, the $(p+3)$ convergence condition (3.2b) can be rewritten as (3.7) in term of Θ_k. □

Note that, instead of (3.4), (3.6) can be used parameterized formula (2.16). In this case, we have a family of iterations (3.1) with free parameters $\alpha, a, b, d \in \mathbb{R}$. Another iterations with eight-order of convergence can be constructed by using condition (3.2c) $(p+4)$ order of convergence. Now, we consider this case in more details. Assume that T_k satisfies

$$T_k = \mathbf{1} + \Theta_k + r_k + \beta\Theta_k^2 + \gamma\Theta_k^3 + \dots, \quad \beta, \gamma \in \mathbb{R}. \tag{3.8}$$

Of course, it is easy to show that T_k defined by (2.8) or (2.16) can be rewritten as (3.8) with some constants β and γ. Our task is to find easily implementation algorithm for (3.2c) as (3.7). To do this we first express A_k, B_k and D_k in term of η_k, d_k and C_k. Using Taylor expansion of $F(w_k)$ at point x_k, we have

$$[x_k, w_k; F] = F'(x_k)(I + A_k + B_k + D_k) + O(h^4). \tag{3.9}$$

From (3.9) immediately follows that

$$\begin{aligned} [x_k, w_k; F]^{-1}F'(x_k) &= (I + A_k + B_k + D_k)^{-1} \\ &= I - A_k - B_k - D_k + A_k^2 + 2A_kB_k - A_k^3 + O(h^4). \end{aligned} \tag{3.10}$$

Using (3.9), we have

$$A_k = \gamma F'(x_k)\eta_k = \Delta(I - A_k - B_k - D_k + A_k^2 + 2A_kB_k - A_k^3)\eta_k + O(h^5), \tag{3.11}$$

or

$$A_k = \frac{\Delta\eta_k}{I + \Delta\eta_k}(I - B_k + A_k^2) + O(h^4), \tag{3.12}$$

$$A_k^2 = \left(\frac{\Delta\eta_k}{I + \Delta\eta_k}\right)^2 + O(h^4) = \Delta^2\eta_k^2(I - 2\Delta\eta_k) + O(h^4), \tag{3.13}$$

$$A_k^3 = \Delta^3\eta_k^3 + O(h^4).$$

Analogously, we have

$$D_k = \frac{\Delta^3}{4}C_k + O(h^4), \tag{3.14}$$

and

$$B_k = -(\gamma F'(x_k))^2 d_k = -\Delta^2(I - 2A_k)d_k + O(h^4). \tag{3.15}$$

If we take (3.13) and (3.15) into account in (3.11) then we get

$$A_k = \Delta\eta_k - \Delta^2\eta_k^2 + 2\Delta^3\eta_k^3 + \Delta^3\eta_k d_k + O(h^4). \tag{3.16}$$

So (3.15) becomes as

$$B_k = -\Delta^2 d_k + 2\Delta^3 \eta_k + O(h^4). \tag{3.17}$$

Substituting (3.14), (3.16) and (3.17) into (3.2c), we get

$$\begin{aligned} H_k =& I + (2+\Delta)\eta_k + (6+2\Delta-\Delta^2)\eta_k^2 + (3-\Delta^2)d_k + (3\Delta^3 - 2\Delta^2 \\ &+ 3\Delta + 20)d_k\eta_k + (2\Delta^3 - 2\Delta^2 + 6\Delta + 20)\eta_k^3 + \big(1 + \frac{\Delta^3}{4}\big)C_k + O(h^4). \end{aligned} \tag{3.18}$$

Now, we will establish the connection between η_k and Θ_k. Using Taylor expansion of smooth enough vector function $F(y_k), F(z_k)$ and after some manipulations we obtain

$$F'(x_k)^{-1}F'(y_k) = I - 2\eta_k(I - A_k - B_k + A_k^2) - 3d_k(I - 2A_k) - C_k + O(h^4), \tag{3.19}$$

and

$$\begin{aligned} \Theta_k = \frac{F(y_k)}{F(x_k)} =& A_k + B_k - A_k^2 + \eta_k(I - 2A_k) + d_k(I - 3A_k) + D_k \\ &+ A_k^3 - 2A_kB_k + (3A_k^2 - 2B_k)\eta_k + \frac{1}{4}C_k + O(h^4), \end{aligned} \tag{3.20}$$

and

$$\frac{F(z_k)}{F(y_k)} = I - T_kS + T_k^2P + O(h^4), \tag{3.21}$$

where

$$\begin{aligned} S &= F'(x_k)^{-1}F'(y_k)[x_k, w_k; F]^{-1}F'(x_k), \\ P &= \frac{1}{2}F'(x_k)^{-1}F''(y_k)([x_k, w_k; F]^{-1}F'(x_k))^2F'(x_k)^{-1}F(y_k). \end{aligned} \tag{3.22}$$

It is easy to show that using (3.14), (3.16) and (3.17) in (3.20), we get

$$\begin{aligned} \eta_k =& \frac{\Theta_k}{1+\Delta} + (\Delta - 1)d_k + \frac{2\Delta}{(1+\Delta)^2}\Theta_k^2 + \frac{3\Delta^2}{(1+\Delta)^3}\Theta_k^3 \\ &- \Delta d_k\Theta_k - \frac{\Delta^2 - \Delta + 1}{4}C_k + O(h^4). \end{aligned} \tag{3.23}$$

As a consequence of (3.23) we have

$$\begin{aligned} \eta_k^2 &= \frac{\Theta_k^2}{(1+\Delta)^2} + \frac{2(\Delta-1)}{1+\Delta}\Theta_k d_k + \frac{4\Delta}{(1+\Delta)^3}\Theta_k^3 + O(h^4), \\ \eta_k^3 &= \frac{\Theta_k^3}{(1+\Delta)^3} + O(h^4). \end{aligned} \tag{3.24}$$

Using (3.10), (3.19) in (3.22), we get

$$S = I - A_k - B_k + A_k^2 - 2\eta_k + 4\eta_kA_k - 3d_k + O(h^4), \tag{3.25}$$

and

$$P = \eta_k[A_k + B_k - A_k^2 + \eta_k(I - 2A_k) + d_k](I - 2A_k) + 3d_k(A_k + \eta_k) + O(h^4), \tag{3.26}$$

in which we have used $F''(y_k) = F''(x_k) + F'''(x_k)(y_k - x_k) + O(h^2)$. By virtue of (3.16), (3.17) and (3.23), one can rewrite (3.25) and (3.26) as:

$$S = I - \frac{2+\Delta}{1+\Delta}\Theta_k - (\Delta+1)d_k + \Delta d_k\Theta_k + \frac{\Delta^2 - \Delta - 2}{4}C_k + O(h^4), \tag{3.27}$$

and

$$P = \frac{\Theta_k^2}{1+\Delta} + (\Delta+2)\Theta_k d_k + O(h^4). \tag{3.28}$$

Now, using (3.23) in (3.18) and (3.8), (3.27), (3.28) in (3.21), we obtain

$$\begin{aligned} H_k =& I + \frac{2+\Delta}{1+\Delta}\Theta_k + (\Delta+1)d_k + \frac{6+6\Delta+\Delta^2}{(1+\Delta)^2}\Theta_k^2 + (\Delta+8)\Theta_k d_k \\ &+ \frac{\Delta^3+12\Delta^2+30\Delta+20}{(1+\Delta)^3}\Theta_k^3 - \frac{\Delta^2-\Delta-2}{4}C_k + O(h^4), \end{aligned} \tag{3.29}$$

LEMMA 1. *Let $F(x)$ be five times continuous differentiable function on D. Then holds*

$$\begin{aligned} [F'(x_k)]^{-1}F'(y_k) = I - &\frac{2+\Delta}{1+\Delta}\Theta_k - (\Delta+1)d_k \\ &+3\Delta\Theta_k d_k + \frac{\Delta^2-\Delta-2}{4}C_k + O(h^4). \end{aligned} \tag{3.30}$$

Proof. Using Taylor expansion of $F'(y_k)$ at point x_k, we have

$$\begin{aligned} F'(y_k) = F'(x_k) + F''(x_k)[y_k - x_k] + &\frac{F'''(x_k)}{2}(y_k - x_k)^2 \\ &+ \frac{F'''(x_k)}{3!}(y_k - x_k)^3 + O(h^4), \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} y_k - x_k &= -[x_k, w_k; F]^{-1}F(x_k) = -(I + A_k + B_k + D_k)^{-1}\xi_k, \\ \xi_k &= [F'(x_k)]^{-1}F(x_k). \end{aligned} \tag{3.32}$$

Using relation (3.23) on (3.16), we obtain

$$\begin{aligned} A_k =& \frac{\Delta}{1+\Delta}\Theta_k + \frac{\Delta^2}{(1+\Delta)^2}\Theta_k^2 + \Delta(\Delta-1)d_k + \frac{\Delta^3}{(1+\Delta)^3}\Theta_k^3 \\ &+ \frac{\Delta^2(1-2\Delta)}{1+\Delta}\Theta_k d_k - \frac{\Delta(\Delta^2-\Delta+1)}{4}c_k + O(h^4). \end{aligned} \tag{3.33}$$

Using (3.14), (3.15) and (3.33), we get

$$\begin{aligned} (I + A_k + B_k + D_k)^{-1} &= I - A_k - B_k - D_k + A_k^2 + 2A_kB_k - A_k^3 + O(h^4) \\ &= I - \frac{\Delta}{1+\Delta}\Theta_k + \Delta d_k - \frac{3\Delta^2}{1+\Delta}\Theta_k d_k \\ &\quad + \frac{\Delta(1-\Delta)}{4}c_k + O(h^4). \end{aligned} \tag{3.34}$$

From (3.31), it follows that

$$[F'(x_k)]^{-1}F'(y_k) = I - 2\eta_k(I - A_k - B_k + A_k^2) - 3d_k(I - 2A_k) - C_k + O(h^4), \tag{3.35}$$

in which we have used (3.32) and (3.34). Substituting (3.23), (3.33) and

$$B_k = -\Delta^2 d_k + 2\Delta^3\eta_k d_k = -\Delta^2 d_k + \frac{2\Delta^3}{1+\Delta}\Theta_k d_k + O(h^4), \tag{3.36}$$

into (3.35), we get (3.30). □

LEMMA 2. *Let $F(x)$ be five-times continuously differentiable function on D. Then holds*

$$\begin{aligned}\frac{F(z_k)}{F(y_k)} = &\left(\left(\frac{2+\Delta}{1+\Delta}\right)^2 + \frac{1}{1+\Delta} - \beta\right)\Theta_k^2 + (\Delta+1)d_k \\ &+ \left(\beta\frac{2+\Delta}{1+\Delta} - \gamma + \frac{2(2+\Delta)}{(1+\Delta)^2}\right)\Theta_k^3 \\ &+ (4-\Delta)\Theta_k d_k - \frac{\Delta^2-\Delta-2}{4}c_k + O(h^4).\end{aligned} \tag{3.37}$$

Proof. Using (3.30), (3.34) in (3.22), we obtain

$$S_k = I - \frac{2+\Delta}{1+\Delta}\Theta_k - (\Delta+1)d_k + 3\Delta\Theta_k d_k + \frac{\Delta^2-\Delta-2}{4}c_k + O(h^4), \tag{3.38}$$

and

$$P_k = \frac{1}{1+\Delta}\Theta_k^2 + (\Delta+2)\Theta_k d_k + O(h^4), \tag{3.39}$$

Substituting (3.8), (3.38) and (3.39) into (3.21), we immediately obtain (3.37). □

Note that in proof of lemma 1 and 2, we essentially use the permutation properties of $q-$ linear operators [3]. Now we are ready to prove the following theorem.

THEOREM 6. *The local convergence order of iteration* (3.1) *equal to eight, if and only if the parameter T_k satisfies condition* (3.8) *and H_k satisfies*

$$H_k = T_k + \Theta_k r_k + (\Theta_k + r_k)((\beta-1)\Theta_k^2 - r_k^2) + (1 + 2(\Theta_k + r_k))S_k \tag{3.40}$$

where

$$\Theta_k = \frac{F(y_k)}{F(x_k)}, \quad r_k = \frac{F(y_k)}{F(w_k)}, \quad S_k = \frac{F(z_k)}{F(y_k)}.$$

Proof. From (3.29) and (3.37), it follows that

$$\begin{aligned}H_k - \frac{F(z_k)}{F(y_k)} =& 1 + \frac{2+\Delta}{1+\Delta}\Theta_k + \left(\beta + \frac{1}{1+\Delta}\right)\Theta_k^2 + \\ &\left(\gamma - \beta\frac{2+\Delta}{1+\Delta} + \frac{\Delta^3+10\Delta^2+24\Delta+16}{(1+\Delta)^3}\right)\Theta_k^3 \\ &+ 2(2+\Delta)\Theta_k d_k + O(h^4),\end{aligned} \tag{3.41}$$

or

$$\begin{aligned}H_k =& \mathbf{1} + \frac{2+\Delta}{1+\Delta}\Theta_k + \left(\beta + \frac{1}{1+\Delta}\right)\Theta_k^2 + \left(r - \beta\frac{2+\Delta}{1+\Delta}\right. \\ &\left.+ \frac{(\Delta+2)(\Delta^2+8\Delta+8)}{(1+\Delta)^3}\right)\Theta_k^3 + 2(2+\Delta)\Theta_k d_k + \frac{F(z_k)}{F(y_k)} + O(h^4).\end{aligned} \tag{3.42}$$

From (3.37), we find that

$$d_k = \frac{1}{1+\Delta}\left(\frac{F(z_k)}{F(y_k)} + \left(\beta - \frac{1}{1+\Delta} - \left(\frac{2+\Delta}{1+\Delta}\right)^2\right)\Theta_k^2\right) + O(h^3). \tag{3.43}$$

Table 2: For ρ

T_k	66	(2.35)	(3.8)	(2.35)	(3.8)
H_k	66	(3.6)	(3.45)	(3.7)	(3.40)
ρ	6	7		8	

Substituting (3.43) into (3.42) we obtain

$$\begin{aligned} H_k =& \mathbf{1} + \frac{2+\Delta}{1+\Delta}\Theta_k + \left(\beta + \frac{1}{1+\Delta}\right)\Theta_k^2 \\ &+ \left(\gamma + \beta\frac{2+\Delta}{1+\Delta} - \frac{2+\Delta}{1+\Delta}\frac{\Delta^2+2\Delta+2}{(1+\Delta)^2}\right)\Theta_k^3 \\ &+ \left(1 + 2\frac{2+\Delta}{1+\Delta}\Theta_k\right)\frac{F(z_k)}{F(y_k)} + O(h^4). \end{aligned} \tag{3.44}$$

Using (2.14) in (3.44), we get (3.40). □

The formulas (3.8) and (3.44) are extension of formulas (3.23) and (3.24) in [8] to multidimensional case. When $\Delta = 0$ the formulas (3.8) and (3.40) lead to

$$T_k = 1 + 2\Theta_k + \beta\Theta_k^2 + \gamma\Theta_k^3 + \cdots,$$

and

$$H_k = 1 + 2\Theta_k + (\beta+1)\Theta_k^2 + (\gamma + 2\beta - 4)\Theta_k^3 + (1 + 4\Theta_k)\frac{F(z_k)}{F(y_k)} + O(h^4),$$

which are the necessary and sufficient conditions for eighth-order derivative-based three-step iterations [9].

Remark 5. *If we ignore the $O(h^3)$ terms in* (3.44), *we obtain*

$$H_k = \mathbf{1} + \Theta_k + r_k + \beta\Theta_k^2 + \Theta_k r_k + \frac{F(z_k)}{F(y_k)} + O(h^3), \tag{3.45}$$

which is (together with (3.8)*) sufficient seventh-order convergence condition [11] for iterations* (3.1).

Summarizing the results of theorems we present in Table 2 the order ρ of convergence of iterations (3.1) and corresponding choices of parameters T_k and H_k.

Remark 6. *The proposed iterations* (2.1) *and* (3.1) *with vector coefficients require only one matrix inversion at each iteration step. So they belong to the class of the best iterations with high efficiency indices [13].*

4. Construction and extension of higher-order derivative-free iterations

In order to construct concrete iterations (3.1) we use, besides of (2.23), the following easily verifying relations for divided difference:

$$F[y_k,z_k]=\frac{[x_k,w_k;F]}{T_k}(1-S_k),\quad \frac{F(w_k)}{F(x_k)}=(1+\Delta)(\mathbf{1}+O(h^3)), \tag{4.1a}$$

$$F[x_k,z_k]=[x_k,w_k;F]\frac{1-\dfrac{F(z_k)}{F(x_k)}}{1+T_k\Theta_k},\ F[y_k,w_k]=[x_k,w_k;F]\big(1-\frac{\Theta_k}{1+\Delta}\big)+O(h^3), \tag{4.1b}$$

$$F[x_k,w_k]=[x_k,w_k;F](1+O(h^3)), \tag{4.1c}$$

$$F[z_k,w_k]=[x_k,w_k;F]\left(1-\frac{1}{1+\Delta}\Theta_k-\frac{1}{1+\Delta}\Theta_k^2\right)+O(h^3).$$

We assume that T_k in the second step of (3.1) satisfies the condition (2.8) that guarantees $F(z_k)=O(h^4)$. Let H_k in (3.1) be a form

$$H_k=\frac{1+A\Theta_k+B\Theta_k^2+wS_k}{[x_k,w_k;F]^{-1}([aF[x_k,z_k]+bF[z_k,y_k]+CF[x_k,y_k])},\quad a+b+c=1,\ \ a,b,c\in R, \tag{4.2}$$

where

$$A=(1-b)(\hat{d}_k-1),\ B=(\beta-\hat{d}_k)(1-b)+\frac{1-a}{1+\Delta},\ w=1-b,\ \hat{d}_k=\frac{2+\Delta}{1+\Delta}. \tag{4.3}$$

Theorem 7. *The iteration* (3.1) *with* (2.8) *and* (4.2) *has the order of convergence seven.*

Proof. Using formulas (4.1) it is easy to show that

$$\begin{aligned}[x_k,w_k;F]^{-1}(aF[x_k,z_k]+bF[z_k,y_k]+cF[x_k,y_k])&=1+(a+b+b\hat{d}_k)\Theta_k\\&+[a(\hat{d}_k-1)+b(\beta-\hat{d}_k^2)+(a+c+b\hat{d}_k)]\Theta_k+bS_k+O(h^3).\end{aligned} \tag{4.4}$$

The comparison of the seventh-order convergence condition (3.45) and (4.2), (4.4) gives us

$$\begin{aligned}1+\hat{d}_k\Theta_k+(\beta+\frac{1}{1+\Delta})\Theta_k^2+S_k&=1+(A+a+b+c+b\hat{d}_k)\Theta_k+\{B+a(\hat{d}_k-1)\\&+b(\beta-\hat{d}_k^2)+(a+c+b\hat{d}_k)^2+A(a+c+b\hat{d}_k)\}\Theta_k+(w+b)s,\end{aligned}$$

that holds for (4.3). □

Note that formula (4.2) is an extension of formula (32) in [21] to the multidimensional case.

Remark 7. *If we ignore the terms with order* $O(h^2)$ *in* (4.2) *then* (4.2) *leads to*

$$H_k=\frac{1}{[x_k,w_k;F]^{-1}}\frac{1+\frac{1-b}{1+\Delta}\Theta_k}{(aF[z_k,x_k]+bF[z_k,x_k]+cF[x_k,y_k])},\ a+b+c=1. \tag{4.5}$$

In this case, the convergence order of (3.1) reduced by one unit. The iteration (3.1) with (2.8) and (4.5) has order of convergence six. We consider some particular cases of sixth-order methods.

(1) Let $a = b = 0,\ c = 1$ Then (4.5) leads to

$$\begin{aligned} H_k =& \frac{1 + \frac{1}{1+\Delta}\Theta_k}{1 - \Theta_k} = \frac{\mathbf{1} + \dfrac{F(y_k)}{F(w_k)}}{\mathbf{1} - \dfrac{F(y_k)}{F(x_k)}} \\ =& \mathbf{1} + \frac{F(y_k)}{F(x_k)} + \frac{F(y_k)}{F(w_k)} + O(h^2) = T_k + O(h^2). \end{aligned} \tag{4.6}$$

(2) Let $a = 1,\ b = c = 0$. Then (4.5) leads to

$$H_k = \frac{[x_k, w_k; F]}{F[x_k, z_k]}\left(1 + \frac{F(y_k)}{F(w_k)}\right). \tag{4.7}$$

(3) $a = c = 0,\ b = 1$. Then (4.5) leads to

$$H_k = \frac{1}{[x_k, w_k; F]^{-1}F[z_k, y_k]}. \tag{4.8}$$

(4) The linear combination of (4.7) and (4.8) gives

$$H_k = \frac{[x_k, w_k; F]}{a + b}\left[\frac{a}{F[y_k, z_k]} + \frac{b(1 + \dfrac{F(y_k)}{F(w_k)})}{F(x_k, z_k)}\right]. \tag{4.9}$$

If we take into account $1 + \dfrac{F(y_k)}{F(w_k)} = \dfrac{1 - 2r_k}{1 - 3r_k} + O(h^2)$ then (4.9) can be rewritten as

$$H_k = \frac{[x_k, w_k; F]}{a + b}\left[\frac{a}{F[y_k, z_k]} + \frac{1 - 2r_k}{1 - 3r_k}\frac{b}{F[x_k, z_k]}\right]. \tag{4.10}$$

When $a = b$ then (4.10) leads to

$$H_k = \frac{[x_k, w_k; F]}{2}\left[\frac{1}{2} + \frac{1 - 2r_k}{1 - 3r_k}\frac{1}{F[x_k, z_k]}\right]. \tag{4.11}$$

The iteration (3.1) with H_k given by (4.11) can be considered as extension of $(FD4)$ iteration [20] to multidimensional case. Now, we propose some concrete choice of H_k that guarantees seven-order of iteration (3.1).

(1) Let $a = c = 0, b = 1$. Then (4.2) becomes

$$\begin{aligned} H_k =& \frac{1}{[x_k, w_k; F]^{-1}F[z_k, y_k]}(1 + \frac{1}{1 + \Delta}\Theta_k^2) \\ =& \frac{1}{[x_k, w_k; F]^{-1}F[z_k, y_k]}\left(1 + \frac{F(y_k)}{F(x_k)}\frac{F(y_k)}{F(w_k)}\right). \end{aligned} \tag{4.12}$$

(2) $b = 2,\ a = -1\ c = 0$. Then (4.2) leads to

$$\begin{aligned} H_k =& \frac{1}{[x_k, w_k; F]^{-1}(2F[z_k, y_k] - F[z_k, x_k])} \\ & \left(1 - \frac{F(y_k)}{F(w_k)} + (\hat{d}_k - \beta)\left(\frac{F(y_k)}{F(x_k)}\right)^2 + 2\frac{F(y_k)}{F(x_k)}\frac{F(y_k)}{F(w_k)} - \frac{F(z_k)}{F(y_k)}\right). \end{aligned} \tag{4.13}$$

$2'$. Let $a=1$ and $b=c=0$. Then (4.2) can be expressed as

$$H_k = \frac{\mathbf{1} + \frac{F(y_k)}{F(w_k)} + (\beta - \hat{d}_k)\left(\frac{F(y_k)}{F(x_k)}\right)^2 + \frac{F(z_k)}{F(y_k)}}{[x_k, w_k; F]F[x_k, z_k]}. \tag{4.14}$$

$2''$. It is easy to check that from (4.14). It follows that see [21].

$$H_k = \left(1 - (\hat{d_k^2} - \beta)\frac{F(y_k)}{F(w_k)}\left(\frac{F(y_k)}{F(x_k)}\right)\right)\frac{[x_k, w_k; F]F(x_k, y_k)}{F[x_k, z_k]F[z_k, y_k]}. \tag{4.15}$$

If we choose T_k as

$$T_k = \frac{1}{1 - \hat{d}_k\Theta_k}, \tag{4.16}$$

then $\beta = d_k^2$ and (4.15) converted to

$$H_k = \frac{[x_k, w_k; F]\cdot F[x_k, y_k]}{F[x_k, z_k]\cdot F[z_k, y_k]}. \tag{4.17}$$

In this case, the iteration (3.1) with (4.16), (4.17) is indeed derivative-free variant of reduced scheme given in [22] to multidimensional case.

(3) $a=b=1,\ c=-1$. Then (4.2) leads to

$$H_k = \frac{1}{[x_k, w_k; F]^{-1}(F[z_k, x_k] + F[z_k, y_k] - F[x_k, y_k])}. \tag{4.18}$$

The iteration (3.1) with (4.18) can be considered as extension of $(FD7)$ in [20].

(4) The seventh-order convergence condition (3.45) can be written as

$$H_k = T_k + \frac{F(y_k)}{F(x_k)}\frac{F(y_k)}{F(w_k)} + \frac{F(z_k)}{F(y_k)} \tag{4.19}$$

(5) Let H_k in (3.1) is given by

$$\begin{aligned} H_k =& \frac{[x_k, w_k; F]}{aF[z_k, x_k] + (1-a)F[z_k, w_k]} \\ & \left\{1 + \frac{F(z_k)}{F(y_k)} + a\frac{F(y_k)}{F(w_k)} + (1-a)\frac{F(y_k)}{F(x_k)} + w_1\left(\frac{F(y_k)}{F(w_k)}\right)^2 + w_2\left(\frac{F(y_k)}{F(x_k)}\right)^2\right\}, \\ & w_2 = \beta - \frac{(1+a\Delta)(2+\Delta) + w_1}{(1+\Delta)^2}. \end{aligned} \tag{4.20}$$

Using the relations (4.1) it is easy show that H_k given by (4.19) satisfies the condition (3.45). The iteration (3.1) with H_k given by (4.19) with $a=1,\ w_1=0$ and $a=0, w_2=0$ can be considered as extension of $(FS3-1)$ and $(FS4-1)$ given in [19] Note that in $FS3-1$ and $(FS4-1)$ used T_k given by (2.22c), whereas one extension used T_k given by (2.22a)-(2.22b).

As a consequence, the theorem 6 proved in [21] holds true in multidimensional case. That is the iteration (3.1) with (3.8) and H_k given by

$$H_k = \frac{F[x_k, w_k](1 + A\Theta_k + B\Theta_k^2 + C\Theta_k^3 + (w + \delta\Theta_k)S_k)}{aF[x_k, z_k] + bF[z_k, y_k] + cF[x_k, y_k]},\quad a+b+c=1, \tag{4.21}$$

Table 3: For ρ

T_k	(2.22)	(2.22)
H_k	(4.6),(4.7),(4.8),(4.9),(4.10)	(4.12),(4.13),(4.18),(4.19),(4.20)
ρ	6	7

has the order of convergence eight, if the following conditions hold:

$$\begin{aligned} &A = (1-b)(\hat{d}_k - 1), \quad B = (\beta - \hat{d}_k)(1-b) + \frac{1-a}{1+\Delta}, \\ &w = 1-b, \quad \delta = -a + b - 1 + \hat{d}_k(2-b), \quad \hat{d}_k = \frac{2+\Delta}{1+\Delta}, \\ &C = a - 1 + (b-a)\beta + (1-b)\gamma + \frac{a+b-2}{1+\Delta} + \frac{c-2}{(1+\Delta)^2} - \frac{1}{(1+\Delta)^3}. \end{aligned} \tag{4.22}$$

Thus, we have a family of eighth -order iterations (3.1) with (3.8) and (4.21).

The implementation algorithm is given by (4.21), in which we used:

$$\begin{aligned} &A\Theta_k = (1-b)r_k, \quad B\Theta_k^2 = \Big((1-b)(\beta-1)\Theta_k + (b-a)r_k\Big)\Theta_k, \\ &w = 1-b, \quad \delta\Theta_k = (1-a)\Theta_k + (2-b)r_k, \\ &C\Theta_k^3 = [(a-1) + (b-a)\beta + (1-b)\gamma]\Theta_k^3 \\ &\qquad + (a+\beta-2)\Theta_k^2 r_k + (c-2)\Theta_k r_k^2 - r_k^3. \end{aligned} \tag{4.23}$$

Most easy particular cases of (4.21) are shown in Table 4.

Table 4

a	b	c
0	0	1
0	1	0
1	0	0

Another form H_k satisfying the condition (3.44) is [21]

$$H_k = \frac{1}{\mathbf{1} - \dfrac{F(z_k)}{F(w_k)}} \left(1 + (\hat{d}_k - \beta) r_k \Theta_k^2\right) \frac{[x_k, w_k; F]F[x_k, y_k]}{F[x_k, z_k]F[z_k, y_k]}. \tag{4.24}$$

For example, if we choose T_k such that

$$T_k = \frac{\mathbf{1}}{1 - \hat{d}_k \Theta_k}, \tag{4.25}$$

then $\beta = \hat{d}_k^2$ and the formula (4.24) becomes as

$$H_k = \frac{1}{1 - \dfrac{F(z_k)}{F(w_k)}} \frac{[x_k, w_k; F]F[x_k, y_k]}{F[x_k, z_k] \cdot F[z_k, y_k]}. \tag{4.26}$$

The iteration (3.1) with T_k and H_k given by (4.25) and (4.26) can be considered as multidimensional of eighth-order Sharma et al. method [22] for solving nonlinear equations.

5. Numerical experiments and efficiency index

We demonstrate the efficiency of the proposed methods and verify the corresponding theoretical results. The methods examined include the fourth-order methods M4, the sixth-order methods M6, the seventh-order methods M7, and the eighth-order methods M8. All of our numerical computations were performed in programming system MATHEMATICA 14.1 using multi-precision arithmetic with 1000 digits for Intel(R) i5-13700, 2.10GHz, 32GB RAM. In all experiments, the iterative process was terminated when the following stopping criterion was satisfied:

$$\|x_{k-1} - x_k\| \leq 10^{-60}.$$

The computational order of convergence (ACOC) at iteration k is defined by:

$$\rho_k = \frac{\log\left(\|F(x_{k+1})\|/\|F(x_k)\|\right)}{\log\left(\|F(x_k)\|/\|F(x_{k-1})\|\right)}.$$

Furthermore, the methods with orders 4, 6, 7, and 8 in Table 5 are denoted by M4, M6, M7, and M8, respectively. For the numerical experiments, we consider the following test problem:

Table 5: Methods

Methods	T_k	H_k	ρ
(2.1)	$\mathbf{1} + \Theta_k + r_k$	-	4
(3.1)	$\mathbf{1} + \Theta_k + r_k$	$\mathbf{1} + \Theta_k + r_k$	6
(3.1)	$\mathbf{1} + \Theta_k + r_k$	$T_k + \Theta_k r_k + S_k$	7
(3.1)	$\mathbf{1} + \Theta_k + r_k$	$T_k + \Theta_k r_k + (\Theta_k + r_k)((\beta-1)\Theta_k^2 - r_k^2) + (1 + 2(\Theta_k + r_k))S_k$	8

Example 1. *Consider the nonlinear system:*

$$x_i^2 x_{i+1} - 1 = 0,\ i = 1, 2, \ldots, n-1,$$
$$x_n^2 x_1 - 1 = 0.$$

The exact solution is $(1, 1, \ldots, 1)^T$ *and the initial approximation is* $x_0 = (1.25, 1.25, \cdots, 1.25)^T$.

Example 2. *Consider the system of equations:*

$$\begin{cases} x_i x_{i+1} - e^{-x_i} - e^{-x_{i+1}} = 0, & 1 \leq i \leq n-1, \\ x_n x_1 - e^{-x_n} - e^{-x_1} = 0. \end{cases}$$

Starting value is $x_0 = (1, 1, 1, 1, \ldots, 1, 1)^T$ *and the approximate solution is* $x^* \approx (0.901, 0.901, \ldots, 0.901)^T$.

Example 3. *We consider the following system of nonlinear equations involving trigonometric functions:*

$$\begin{cases} x_i \sin(x_{i+1}) - 1 = 0, \; i = 1, 2, \ldots, n-1 \\ x_n \sin(x_1) - 1 = 0 \end{cases}$$

The close-form solution is $(1.11415714, 1.11415714, \ldots, 1.11415714)^T$ *and* $x_0 = (1.3, 1.3, \cdots, 1.3)^T$ *is the initial vector.*

Table 6: Numerical Results for Example 1 (Problem Size n=500)

Method	CPU Time	k	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
M4	26.422	5	3.36×10^{-39}	3.80×10^{-156}	4.00
M6	20.656	4	1.72×10^{-32}	4.89×10^{-194}	6.00
M7	21.094	4	1.25×10^{-67}	1.13×10^{-472}	7.00
M8	21.188	4	3.27×10^{-75}	1.84×10^{-600}	8.00

Table 7: Numerical Results for Example 2 (Problem Size n=500)

Method	CPU Time	k	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
M4	77.875	4	6.32×10^{-65}	3.37×10^{-261}	4.00
M6	56.657	3	2.57×10^{-35}	1.59×10^{-214}	6.00
M7	57.063	3	1.43×10^{-51}	1.78×10^{-364}	7.00
M8	57.796	3	2.84×10^{-62}	6.40×10^{-502}	8.00

Table 8: Numerical Results for Example 3 (Problem Size n=500)

Method	CPU Time	k	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
M4	60.203	4	5.28×10^{-47}	9.20×10^{-191}	4.00
M6	58.796	4	6.39×10^{-141}	1.51×10^{-849}	6.00
M7	44.375	3	3.95×10^{-34}	2.77×10^{-246}	7.00
M8	57.532	3	3.79×10^{-46}	6.32×10^{-256}	8.00

The numerical results are consistent with the theoretical findings presented in the preceding sections. The tables show that M8 is not only faster but also more accurate than the considered methods of orders $\rho = 4, 6, 7$ and 8. From Table 11, we observe that the proposed algorithms are significantly faster than the compared methods. Tables 10 and 11 show that the proposed methods still converge when the initial approximation is located at distances 10 and 100 times greater. This highlights the robustness and wide convergence domain of the methods. To compare different methods, we use the computational efficiency index $CI = \rho^{\frac{1}{C}}$, where ρ is the order of convergence and C is the computational cost of each method. Table 12 presents the index CI of the methods considered. Figure 1 also illustrates the index CI for systems of size ranging from 10 to 100.

Table 9: Results for Example 2 (Problem Size n=200)

Methods	Initial vector	k	$\|F(x_{k+1})\|$	ACOC
M4	x_0	4	3.37×10^{-261}	4.00
	$10x_0$	9	6.50×10^{-78}	4.00
	$100x_0$	42	1.13×10^{-82}	4.00
M6	x_0	3	1.92×10^{-214}	6.00
	$10x_0$	8	8.78×10^{-146}	6.00
	$100x_0$	28	5.74×10^{-296}	6.00
M7	x_0	3	1.12×10^{-364}	7.00
	$10x_0$	6	5.03×10^{-262}	7.00
	$100x_0$	28	2.01×10^{-491}	7.00
M8	x_0	3	6.40×10^{-502}	8.00
	$10x_0$	6	5.28×10^{-182}	8.00
	$100x_0$	25	1.30×10^{-789}	8.00

Table 10: Results for Example 3 (Problem Size n=200)

Methods	Initial vector	k	$\|F(x_{k+1})\|$	ACOC
M4	x_0	4	5.82×10^{-191}	4.00
	$10x_0$	6	7.99×10^{-44}	4.00
	$100x_0$	38	8.53×10^{-11}	4.00
M6	x_0	4	9.60×10^{-850}	6.00
	$10x_0$	6	5.56×10^{-167}	6.00
	$100x_0$	12	1.01×10^{-186}	6.00
M7	x_0	3	1.75×10^{-246}	7.00
	$10x_0$	5	1.21×10^{-434}	7.00
	$100x_0$	29	2.66×10^{-211}	7.00
M8	x_0	3	2.89×10^{-306}	8.00
	$10x_0$	4	7.18×10^{-74}	8.00
	$100x_0$	27	3.64×10^{-217}	8.00

Table 11: CPU Time for Example 1

Method	$n = 100$	$n = 200$	ACOC
M4	1.031	3.89	4.00
M6	0.765	3.078	6.00
M7	0.797	3.063	7.00
M8	0.766	3.093	8.00
$M_{6,2}$ [23]	20.297	152.969	6.00
$NM7$ [24]	27.859	217.531	7.00
$PM1$ [25]	14.201	101.725	8.00

6. Conclusions

In this paper, we propose derivative free two- and three-step iterations with high convergence order from four to eight. Some of these iterations are extensions of many well-known iterations to multidimensional case. Thus, we essentially extend their application domain. Main advantages of the proposed schemes are that they require only one matrix inversion and the parameter coefficients are expressed via multiplication and division of known vectors. These two factors allows to increase their efficiency and make algorithms easy to implement. The results of numerical experiments show their high performance and wide convergence domain and confirm convergence order.

Table 12: The computational efficiency

№	methods	ρ	C_i	CI
1	M4	4	$C_1 = \frac{1}{3}n^3 + 4n^2 + \frac{11}{3}n$	$4^{1/C_1}$
2	M6	6	$C_2 = \frac{1}{3}n^3 + 5n^2 + \frac{14}{3}n$	$6^{1/C_2}$
3	M7	7	$C_3 = \frac{1}{3}n^3 + 5n^2 + \frac{26}{3}n$	$7^{1/C_3}$
4	M8	8	$C_4 = \frac{1}{3}n^3 + 5n^2 + \frac{35}{3}n$	$8^{1/C_4}$
5	M2	8	$C_5 = \frac{1}{3}n^3 + 3n^2 + \frac{2}{3}n$	$2^{1/C_5}$

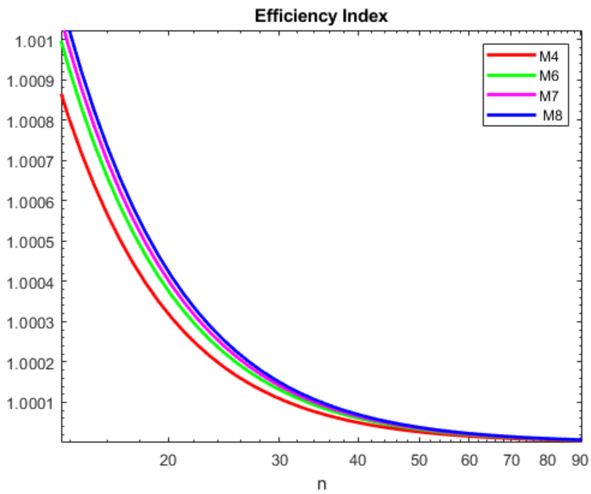

Figure 1: Computational Efficiency Index for $n = 10$ to 100 (logarithmic scale)

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Tugal Zhanlav
Institute of Mathematics and Digital Technology
Mongolian Academy of Sciences
School of Applied Sciences
Mongolian University of Science and Technology
Ulaanbaatar, Mongolia

tzhanlav@yahoo.com
https://orcid.org/0000-0003-0743-5587

Khuder Otgondorj
School of Applied Sciences
Mongolian University of Science and Technology
Ulaanbaatar, Mongolia

otgondorj@must.edu.mn
* *Corresponding author*
https://orcid.org/0000-0003-1635-7971