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Boundedness of Some Hilbert–Type Operators on the Weighted Morrey–Herz Spaces

Tserendorj Batbold, Amarsaikhan Amarbayar, Mario Krnić

Abstract. In this paper we establish necessary and sufficient conditions for the boundedness of a general Hilbert-type operator on the weighted Morrey-Herz spaces, without imposing conditions on a homogeneous kernel. As an application, some particular cases are also considered. Our results are compared with some previously known from the literature.

1. Introduction

Let $K:(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a non-negative measurable homogeneous function of degree $-\lambda$, $\lambda>0$, i.e. $K(tx,ty)=t^{-\lambda}K(x,y)$. The Hilbert-type integral operator

$$Tf(x)=\int_0^\infty K(x,y)f(y)dy, \quad x\geq 0,$$

is one of the most important operators in operator theory and its applications. Actually, numerous classical integral operators are special cases of operator T, for some particular choices of kernel K. For example, we have

– the classical Hilbert integral operator

$$\mathcal{H}f(x)=\int_0^\infty \frac{f(y)}{x+y}dy,$$

for $K(x,y)=\frac{1}{x+y}$

– the Hardy-Littlewood-Polya operator

$$HLf(y)=\int_0^\infty \frac{f(y)}{\max\{x,y\}}dy,$$

for $K(x,y)=\frac{1}{\max\{x,y\}}$

– a generalized Hardy-Littlewood-Polya operator

$$HL_\lambda f(y)=\int_0^\infty \frac{f(y)}{\max\{x^\lambda,y^\lambda\}}dy,$$

for $K(x,y)=\frac{1}{\max\{x^\lambda,y^\lambda\}}$

– the classical Hardy operator

$$Hf(x)=\frac{1}{x}\int_0^x f(y)dy,$$

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for $\lambda = 1, K(x,y) = x^{-1} \cdot \chi_E(x,y), E = \{(x,y) \mid x < y\}$.
The sharp bounds for T on Lebesgue spaces have been studied by Bényi and Oh [2], while the corresponding bounds on the weighted Morrey spaces in multidimensional case have been established by Batbold and Sawano [1]. In 2009, Kuang [5], established the necessary and sufficient conditions for the Hilbert-type operators to be bounded on the weighted Herz spaces under the following conditions on the kernel function:

(C1) There exist constants $C_1(p), C_2(p) > 0$, such that

$$\int_0^\infty t^{\lambda-1-1/q} K(1,t)dt \leq C_1(p) \left(\int_0^\infty t^{(\lambda-1-1/q)p} K^p(1,t)dt \right)^{\frac{1}{p}}, \quad \text{for} \ \ 1 \leq p < \infty$$

and

$$\int_0^\infty t^{\lambda-1-1/q} K(1,t)dt \leq C_2(p) \left(\int_0^\infty t^{(\lambda-1-1/q)p} K^p(1,t)dt \right)^{\frac{1}{p}}, \quad \text{for} \ \ 0 < p < 1.$$

In addition, few years later, Kuang [7], also gave the necessary condition for the boundedness of the Hilbert-type operator on a more general weighted Morrey-Herz spaces, under the following conditions on the kernel function:

(C2) $K(1,t)$ has a compact support on $(0,\infty)$.

(C3) $t^{\lambda-1-(\beta+1)}K(1,t)$ is a concave function on $(0,\infty)$

Recently, Yee and Ho [10], established a necessary condition for the boundedness of the Hilbert-type operator on the Morrey-Herz spaces when $\lambda = 1$. For some related results about the boundedness of Hilbert-type and Hardy-type operators on Morrey-Herz spaces the reader is referred to papers [3]– [7] and [10], as well as to the references cited therein.

Motivated by the above discussed results, our aim in this paper is to establish the boundedness for the Hilbert-type operator T on the weighted Morrey-Herz spaces without imposing conditions (C1)-(C3) on a homogeneous kernel. Hence, our results may be regarded as an extension of the above results. In addition, we will also study some particular Hilbert-type operators.

2. Main Results

At the beginning of this section we will recall a definition of the weighted Morrey-Herz spaces.

Definition 1. *Let $\alpha \in \mathbb{R}, 0 < p < \infty, 0 < q < \infty, \lambda_1 \geq 0$ and let ω be a weight function. The weighted Morrey-Herz space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R},\omega)$ is defined as the space of all functions $f \in L^q_{\text{loc}}(\mathbb{R} \setminus \{0\}, \omega)$ such that*

$$\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R},\omega)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda_1} \left(\sum_{j=-\infty}^{k_0} 2^{j\alpha p} \|f\chi_j\|^p_{L^q_\omega} \right)^{\frac{1}{p}} < \infty.$$

The Morrey-Herz spaces are natural generalizations of the Herz spaces and the central Morrey spaces. For more details about these spaces and their applications in analysis, the reader is referred to [8, 9].

Now, we are ready to state and prove our main result. We write $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+,\omega^*) = M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)$ and $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+,\omega) = M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)$, for brevity.

THEOREM 1. *Let $\alpha, \beta \in \mathbb{R}, \lambda > 0,\ \lambda_1 > 0,\ 0 < p < \infty,\ 1 \le q < \infty,\ \omega(x) = x^{\beta}$ and $\omega^*(x) = x^{(1-\lambda)q+\beta}$. Then the Hilbert-type operator T is bounded from $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)$ to $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)$ if and only if*

$$\int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}} K(1,t)dt < \infty. \tag{2.1}$$

In addition, then holds the inequality

$$\|T\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)\longrightarrow M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} \le C(\alpha,\lambda_1,p)\int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}} K(1,t)dt,$$

where

$$C(\alpha,\lambda_1,p) = \begin{cases} \left(1+2^{|\lambda_1-\alpha|}\right), & 1 \le p < \infty \\ \frac{2^{\lambda_1}}{(2^{\lambda_1 p}-1)^{\frac{1}{p}}}\cdot\left(1+2^{|\lambda_1-\alpha|}\right), & 0 < p < 1 \end{cases}.$$

Proof. (i) Utilizing the Minkowski inequality as well as the change of variables $y = tx$, we have that

$$\begin{aligned}
\|(Tf)\chi_k\|_{L^q_\omega} &= \left\{\int_{A_k}\left|\int_0^\infty K(x,y)f(y)dy\right|^q \omega(x)dx\right\}^{\frac{1}{q}} \\
&= \left\{\int_{A_k}\left|\int_0^\infty K(x,tx)f(tx)xdt\right|^q \omega(x)dx\right\}^{\frac{1}{q}} \\
&= \left\{\int_{A_k}\left|\int_0^\infty x^{1-\lambda}K(1,t)f(tx)dt\right|^q \omega(x)dx\right\}^{\frac{1}{q}} \\
&= \left\{\int_{A_k}\left|\int_0^\infty K(1,t)f(tx)dt\right|^q x^{(1-\lambda)q}\omega(x)dx\right\}^{\frac{1}{q}} \\
&\le \int_0^\infty \left(\int_{A_k}|f(tx)|^q x^{(1-\lambda)q}\omega(x)dx\right)^{\frac{1}{q}} K(1,t)dt \\
&= \int_0^\infty \left(\int_{2^{k-1}t<s\le 2^k t}|f(s)|^q (\frac{s}{t})^{(1-\lambda)q}\omega\left(\frac{s}{t}\right)\frac{1}{t}ds\right)^{\frac{1}{q}} K(1,t)dt \\
&= \int_0^\infty \left(\int_{2^{k-1}t<s\le 2^k t}|f(s)|^q s^{(1-\lambda)q}\omega\left(\frac{s}{t}\right)ds\right)^{\frac{1}{q}} t^{\lambda-1-1/q}K(1,t)dt \\
&= \int_0^\infty \left(\int_{2^{k-1}t<s\le 2^k t}|f(s)|^q s^{(1-\lambda)q+\beta}ds\right)^{\frac{1}{q}} t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt.
\end{aligned}$$

It should be noticed here that for each $t \in (0,\infty)$, there exists an integer m such that $2^{m-1} < t \le 2^m$. Now, set $A_{k,m} = \left\{s \in (0,\infty) : 2^{k+m-1} < s \le 2^{k+m}\right\}$. It is not hard to check that $\{s\ |2^{k-1}t < s \le 2^k t\} \subseteq A_{k-1,m} \cup A_{k,m}$. Hence, we have

$$\begin{aligned}
\|(Tf)\chi_k\|_{L^q_\omega} &\le \int_0^\infty \left(\int_{A_{k-1,m}}|f(s)|^q s^{(1-\lambda)q+\beta}ds + \int_{A_{k,m}}|f(s)|^q s^{(1-\lambda)q+\beta}ds\right)^{\frac{1}{q}} \\
&\quad\times t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt \\
&\le \int_0^\infty \left(\|f\chi_{k+m-1}\|_{L^q_{\omega^*}} + \|f\chi_{k+m}\|_{L^q_{\omega^*}}\right) t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt.
\end{aligned}$$

On the other hand, taking into account the definition of the weighted Morrey-Herz spaces, we obtain

$$\begin{aligned}\|Tf\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} &= \sup_{k_0\in Z} 2^{-k_0\lambda_1}\cdot\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|(Tf)\chi_k\|^p_{L^q_\omega}\right)^{\frac{1}{p}}\\ &\le \sup_{k_0\in Z} 2^{-k_0\lambda_1}\\ &\quad\cdot\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\int_0^\infty (\|f\chi_{k+m-1}\|_{L^q_\omega}+\|f\chi_{k+m}\|_{L^q_\omega})\cdot t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt\right)^p\right)^{\frac{1}{p}}.\end{aligned} \tag{2.2}$$

Now, we have to consider two cases depending on whether $1\le p<\infty$ or $0<p<1$.
Case 1. ($1\le p<\infty$) Using the Minkowski inequality, we have that

$$\begin{aligned}&\|Tf\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)}\\ &\le \sup_{k_0\in Z} 2^{-k_0\lambda_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left\{\int_0^\infty (\|f\chi_{k+m-1}\|_{L^q_\omega}+\|f\chi_{k+m}\|_{L^q_\omega})t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt\right\}^p\right)^{\frac{1}{p}}\\ &\le \int_0^\infty t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)\cdot\left[\sup_{k_0\in Z} 2^{-k_0\lambda_1}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\|f\chi_{k+m-1}\|_{L^q_\omega}+\|f\chi_{k+m}\|_{L^q_\omega}\right)^p\right)^{\frac{1}{p}}\right]dt,\end{aligned}$$

and so

$$\begin{aligned}&\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\cdot\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}(\|f\chi_{k+m-1}\|_{L^q_\omega}+\|f\chi_{k+m}\|_{L^q_\omega})^p\right)^{\frac{1}{p}}\\ &\quad\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\left[\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_{k+m-1}\|^p_{L^q_\omega}\right]^{\frac{1}{p}}+\left[\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_{k+m}\|^p_{L^q_\omega}\right]^{\frac{1}{p}}\right)\\ &\quad= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left(\left[2^{(1-m)\alpha p}\sum_{k=-\infty}^{k_0} 2^{(k+m-1)\alpha p}\|f\chi_{k+m-1}\|^p_{L^q_\omega}\right]^{\frac{1}{p}}\right.\\ &\qquad\left.+\left[2^{-m\alpha p}\sum_{k=-\infty}^{k_0} 2^{(k+m)\alpha p}\|f\chi_{k+m}\|^p_{L^q_\omega}\right]^{\frac{1}{p}}\right)\\ &\quad= 2^{(1-m)\alpha}\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}\cdot 2^{(m-1)\lambda_1}+2^{-m\alpha}\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}\cdot 2^{m\lambda_1}\\ &\quad= \left(2^{(m-1)(\lambda_1-\alpha)}+2^{m(\lambda_1-\alpha)}\right)\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}.\end{aligned}$$

Therefore we get

$$\begin{aligned}\|Tf\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} &\le \int_0^\infty \left(2^{(m-1)(\lambda_1-\alpha)}+2^{m(\lambda_1-\alpha)}\right)t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt\cdot\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}\\ &\le \left(1+2^{|\lambda_1-\alpha|}\right)\int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt\cdot\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}.\end{aligned}$$

Case 2. ($0 < p < 1$ and $\lambda_1 > 0$) In this setting, we have that

$$\begin{aligned}\|f\chi_{k+m-1}\|_{L^q_\omega} &= \left(2^{(k+m-1)\alpha p} \cdot \|f\chi_{k+m-1}\|^p_{L^q_\omega}\right)^{\frac{1}{p}} \cdot 2^{-(k+m-1)\alpha} \\ &\le 2^{(\lambda_1-\alpha)(k+m-1)} \cdot \|f\|_{M\dot{K}^{\alpha\lambda_1}_{p,q}(\omega^*)}.\end{aligned}$$

Further, taking into account this relation as well as inequality (2.2), we obtain

$$\begin{aligned}\|Tf\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \cdot \left[\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\int_0^\infty (2^{(\lambda_1-\alpha)(k+m-1)}\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}\right.\right. \\ &\qquad \left.\left. +2^{(\lambda_1-\alpha)(k+m)}\|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)})t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt\right)^p\right]^{\frac{1}{p}} \\ &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \cdot \left[\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\int_0^\infty (2^{(\lambda_1-\alpha)(k+m-1)}\right.\right. \\ &\qquad \left.\left. +2^{(\lambda_1-\alpha)(k+m)})t^{\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt\right)^p\right]^{\frac{1}{p}} \|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)} \\ &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \cdot \left[\sum_{k=-\infty}^{k_0} 2^{\lambda_1 kp}\right]^{\frac{1}{p}} \cdot \left(1+2^{|\lambda_1-\alpha|}\right) \\ &\qquad \times \int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt \cdot \|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} \\ &= \frac{2^{\lambda_1}}{(2^{\lambda_1 p}-1)^{\frac{1}{p}}} \cdot \left(1+2^{|\lambda_1-\alpha|}\right) \\ &\qquad \times \int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt \cdot \|f\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)}.\end{aligned}$$

(ii) Now, our intention is to show that the integral $\int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt$ converges when operator T is bounded. Hence, suppose that T is bounded from $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)$ to $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)$. Again we have to consider two cases.

Case 1. ($1 \le p < \infty$ and $\lambda_1 = 0$) Let $\varepsilon > 0$ be a sufficiently small number and let

$$f_\varepsilon(x) = \begin{cases} 0, & 0 < x \le 1 \\ x^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}, & x > 1 \end{cases}.$$

Then

$$\begin{aligned}\|f_\varepsilon\lambda_k\|^q_{L^q_{\omega^*}} &= \int_{2^{k-1}<x\le 2^k} x^{(-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon)q} \cdot x^{(1-\lambda)q+\beta}dx \\ &= \int_{2^{k-1}<x\le 2^k} x^{-\alpha q-1-\varepsilon q}dx = \left|\frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q}\right| \cdot 2^{-k(\alpha+\varepsilon)q},\end{aligned}$$

provided that $k \ge 1$, while for $k \le 0$ we have that

$$\|f_\varepsilon\lambda_k\|^q_{L^q_{\omega^*}} = 0.$$

Therefore we obtain

$$\begin{aligned}\|f_\varepsilon\|_{\dot{K}^{\alpha}_{p,q}(\omega^*)} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\cdot \|f\lambda_k\|^p_{L^q_{\omega^*}}\right)^{\frac{1}{p}}\\ &= \left(\sum_{k=1}^{\infty} 2^{k\alpha p}\cdot 2^{-k(\alpha+\varepsilon)p}\cdot \left|\frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q}\right|^{\frac{p}{q}}\right)^{\frac{1}{p}}\\ &= \left|\frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q}\right|^{\frac{1}{q}}\cdot\left(\sum_{k=1}^{\infty} 2^{-k\varepsilon p}\right)^{\frac{1}{p}}\\ &= \left|\frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q}\right|^{\frac{1}{q}}\cdot\left(\frac{1}{2^{\varepsilon p}-1}\right)^{\frac{1}{p}}.\end{aligned}$$

Now, by a straightforward calculation, we have that

$$\begin{aligned}Tf_\varepsilon(x) &= \int_0^\infty K(x,y)f_\varepsilon(y)dy = \int_1^\infty K(x,y)y^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}dy\\ &= \int_{1/x}^\infty K(x,tx)\cdot(tx)^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}\cdot x dt\\ &= x^{-\alpha-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}\int_{1/x}^\infty K(1,t)\cdot t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}dt,\end{aligned}$$

and consequently,

$$\begin{aligned}&\|Tf_\varepsilon\|_{\dot{K}^{\alpha}_{p,q}(\omega)}\\ &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|(Tf_\varepsilon)\lambda_k\|^p_{L^q_\omega}\right\}^{\frac{1}{p}}\\ &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\int_{A_k}\left|\int_{1/x}^\infty K(1,t)t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}dt\right|^q x^{(-\alpha-\frac{1}{q}-\varepsilon)q}dx\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\ &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\int_{A_k}\left|\int_{1/x}^\infty K(1,t)t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}dt\right|^q x^{-\alpha q-1-\varepsilon q}dx\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\ &\geq \left\{\sum_{k=1}^{\infty} 2^{k\alpha p}\left(\int_{A_k}\left|\int_{1/x}^\infty K(1,t)t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}dt\right|^q x^{-\alpha q-1-\varepsilon q}dx\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\ &= \left\{\sum_{k=1}^{\infty} 2^{k\alpha p}\left(\int_{2^{k-1}<x\leq 2^k} x^{-\alpha q-1-\varepsilon q}\left|\int_{1/x}^\infty K(1,t)t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon}dt\right|^q dx\right)^{\frac{p}{q}}\right\}^{\frac{1}{p}}.\end{aligned}$$

Since $\varepsilon \in (0,1)$, there exists a positive integer number l such that $2^{l-1} \leq \frac{1}{\varepsilon} < 2^l$. Thus, we have

$$\begin{aligned}
&\|Tf_\varepsilon\|_{\dot K^{\alpha}_{p,q}(\omega)} \\
&\geq \left\{ \sum_{k=l+1}^{\infty} 2^{k\alpha p} \left(\int_\varepsilon^\infty K(1,t) t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon} dt \right)^p \cdot \left(\int_{2^{k-1}<x\leq 2^k} x^{-\alpha q-1-\varepsilon q} dx \right)^{\frac{p}{q}} \right\}^{\frac{1}{p}} \\
&= \int_\varepsilon^\infty K(1,t) t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon} dt \cdot \left\{ \sum_{k=1}^{\infty} 2^{k\alpha p} \left(\left| \frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q} \right| \cdot 2^{-k\alpha q - k\varepsilon q} \right)^{\frac{p}{q}} \right\}^{\frac{1}{p}} \\
&= \int_\varepsilon^\infty K(1,t) t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon} dt \cdot \left| \frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q} \right|^{\frac{1}{q}} \cdot \left(\sum_{k=l+1}^{\infty} 2^{-k\varepsilon p} \right)^{\frac{1}{p}} \\
&= \int_\varepsilon^\infty K(1,t) t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon} dt \cdot \left| \frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q} \right|^{\frac{1}{q}} \cdot \left(\frac{1}{2^{\varepsilon p}-1} \right)^{\frac{1}{p}} \cdot \frac{1}{2^{l\varepsilon}},
\end{aligned}$$

which means that

$$\|T\|_{\dot K^{\alpha}_{p,q}(\omega^*) \to \dot K^{\alpha}_{p,q}(\omega)} \geq \frac{\|Tf_\varepsilon\|_{\dot K^{\alpha}_{p,q}(\omega)}}{\|f_\varepsilon\|_{\dot K^{\alpha}_{p,q}(\omega^*)}} \geq 2^{-l\varepsilon} \cdot \int_\varepsilon^\infty K(1,t) t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}-\varepsilon} dt.$$

Now, by letting $\varepsilon \to 0+$, it follows immediately from the Fatou lemma that $\int_0^\infty K(1,t) t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}} K(1,t) dt < \infty$.

Case 2. ($0 < p < \infty$ and $\lambda_1 > 0$) We also take

$$f_0(x) = x^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}.$$

- $\lambda_1 \neq \alpha$. Then,

$$\begin{aligned}
\|f_0\chi_k\|^q_{L^q_{\omega^*}} &= \int_{2^{k-1}<x\leq 2^k} x^{(\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q})q} \cdot x^{(1-\lambda)q+\beta} dx \\
&= \int_{2^{k-1}<x\leq 2^k} x^{\lambda_1 q - \alpha q - 1} dx.
\end{aligned}$$

Moreover, it follows that

$$\begin{aligned}
\|f_0\|_{M\dot K^{\alpha,\lambda_1}_{p,q}(\omega^*)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f_0\chi_k\|^p_{L^q_{\omega^*}} \right)^{\frac{1}{p}} \\
&= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \cdot 2^{k(\lambda_1-\alpha)p} \left| \frac{1-2^{(\alpha-\lambda_1)q}}{(\lambda_1-\alpha)q} \right|^{\frac{p}{q}} \right)^{\frac{1}{p}} \\
&= \left| \frac{1-2^{(\alpha-\lambda_1)q}}{(\lambda_1-\alpha)q} \right|^{\frac{1}{q}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda_1 p} \right)^{\frac{1}{p}} \\
&= \left| \frac{1-2^{(\alpha-\lambda_1)q}}{(\lambda_1-\alpha)q} \right|^{\frac{1}{q}} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \cdot \left(\frac{2^{k_0\lambda_1 p}}{1-\frac{1}{2^{\lambda_1 p}}} \right)^{\frac{1}{p}} \\
&= \left| \frac{1-2^{(\alpha-\lambda_1)q}}{(\lambda_1-\alpha)q} \right|^{\frac{1}{q}} \cdot \frac{2^{\lambda_1}}{(2^{\lambda_1 p}-1)^{\frac{1}{p}}}.
\end{aligned}$$

• $\lambda_1 = \alpha$. Then,

$$\begin{aligned}\|f_0\chi_k\|^q_{L^q_{\omega^*}} &= \int_{2^{k-1}<x\le 2^k} x^{(\lambda-1-\frac{1}{q}-\frac{\beta}{q})q}\cdot x^{(1-\lambda)q+\beta}dx\\ &= \int_{2^{k-1}<x\le 2^k} x^{-1}dx = \ln 2,\end{aligned}$$

and we have that

$$\begin{aligned}\|f_0\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f_0\chi_k\|^p_{L^q_{\omega^*}}\right\}^{\frac{1}{p}}\\ &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\cdot(\ln 2)^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\ &= (\ln 2)^{\frac{1}{q}}\cdot\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\right\}^{\frac{1}{p}}\\ &= (\ln 2)^{\frac{1}{q}}\cdot\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\frac{2^{k_0\alpha p}}{1-\frac{1}{2^{\alpha p}}}\right\}^{\frac{1}{p}}\\ &= (\ln 2)^{\frac{1}{q}}\cdot\frac{2^\alpha}{(2^{\alpha p}-1)^{\frac{1}{p}}}.\end{aligned}$$

Consequently, for all λ_1, α, we have

$$\|Tf_0\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} = \int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt\cdot\|f_0\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}.$$

Hence

$$\|T\| \ge \frac{\|Tf_0\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)}}{\|f_0\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)}} = \int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt$$

and so $\int_0^\infty t^{\lambda_1-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}}K(1,t)dt < \infty$. The proof is now completed. □

Now, our intention is to consider some particular cases of the previous theorem. In particular, we have necessary and sufficient conditions for T to be bounded on the Morrey-Herz spaces in the case of $\lambda = 1$ and $\beta = 0$. More precisely, we have the following result.

Corollary 2.1. *Let $\alpha\in\mathbb{R}$, $0<p<\infty$, $1\le q<\infty$ and $\lambda_1>0$. Then the Hilbert-type operator T is bounded from $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ to $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ if and only if*

$$\int_0^\infty t^{\lambda_1-\alpha-\frac{1}{q}}K(1,t)dt < \infty. \tag{2.3}$$

In addition, the following inequality holds

$$\|T\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)\longrightarrow M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)} \le C(\alpha,\lambda_1,p)\int_0^\infty t^{\lambda_1-\alpha-\frac{1}{q}}K(1,t)dt,$$

where $C(\alpha,\lambda_1,p)$ is defined in the statement of Theorem 1.

It should be noticed here that the above result is a completion of result of Yee and Ho [10], with $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$.

By putting $\lambda_1 = 0$, our Theorem 1 reduces to the corresponding result from Kuang (see [5, 7]), with a weaker conditions.

Corollary 2.2. *Let* $\alpha, \beta \in \mathbb{R}, \lambda > 0,\ 1 \le p < \infty,\ 1 \le q < \infty,\ \omega(x) = x^\beta$ *and* $\omega^*(x) = x^{(1-\lambda)q+\beta}$*. Then the Hilbert-type operator* T *is bounded from* $\dot{K}^\alpha_{p,q}(\omega^*)$ *to* $\dot{K}^\alpha_{p,q}(\omega)$ *if and only if*

$$\int_0^\infty t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}} K(1,t)dt < \infty.$$

In addition, the following inequality holds

$$\|T\|_{\dot{K}^\alpha_{p,q}(\omega^*)\longrightarrow \dot{K}^\alpha_{p,q}(\omega)} \le \left(1 + 2^{|\alpha|}\right) \int_0^\infty t^{-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}} K(1,t)dt.$$

Now, by putting $\alpha = 0, p = q, \lambda_1 = \frac{\theta}{q}, 0 < \theta < 1$, we obtain the result that corresponds to central Morrey spaces.

Corollary 2.3. *Let* $\beta \in \mathbb{R}, \lambda > 0,\ 1 \le q < \infty, 0 < \theta < 1$ *and* $\omega(x) = x^\beta,\ \omega^*(x) = x^{(1-\lambda)q+\beta}$*. Then the operator* $T : \dot{B}^{q,\theta}(\omega^*) \longrightarrow \dot{B}^{q,\theta}(\omega)$ *is bounded if and only if*

$$\int_0^\infty t^{\frac{\theta}{q}+\lambda-1-\frac{1}{q}-\frac{\beta}{q}} K(1,t)dt < \infty.$$

In addition, the following inequality holds

$$\|T\|_{\dot{B}^{q,\theta}(\omega^*)\longrightarrow \dot{B}^{q,\theta}(\omega)} \le \left(1 + 2^{\frac{\theta}{q}}\right) \int_0^\infty t^{\frac{\theta}{q}-\alpha+\lambda-1-\frac{1}{q}-\frac{\beta}{q}} K(1,t)dt.$$

Finally, to conclude the paper, we calculate (2.3) with some particular choices of kernel functions. The starting point is the kernel $K(x,y) = \frac{1}{(x+y)^\lambda}$. In this case, Theorem 1 yields the following consequence.

Corollary 2.4. *Let* $\alpha, \beta \in \mathbb{R}, \lambda > 0,\ \lambda_1 > 0,\ 0 < p < \infty,\ 1 \le q < \infty,\ \omega(x) = x^\beta$ *and* $\omega^*(x) = x^{(1-\lambda)q+\beta}$*. Then the Hilbert-type operator* $\mathcal{H}_\lambda$ *is bounded from* $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ *to* $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ *if and only if*

$$\lambda > \alpha - \lambda_1 + \frac{1}{q} + \frac{\beta}{q} > 0.$$

Then,

$$\|\mathcal{H}_\lambda\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*)\longrightarrow M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} \le C(\alpha,\lambda_1,p)\cdot B\left(\lambda_1 - \alpha + \lambda - \frac{1}{q} - \frac{\beta}{q}, \alpha - \lambda_1 + \frac{1}{q} + \frac{\beta}{q}\right),$$

where $C(\alpha,\lambda_1,p)$ *is defined in the statement of Theorem* 1 *and* B *is the usual beta function defined by* $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,\ a,b > 0.$

Further, for the kernel $K(x,y) = \frac{1}{\max\{x^\lambda, y^\lambda\}}$, Theorem 1 reads as follows:

COROLLARY 2.5. *Let* $\alpha, \beta \in \mathbb{R}, \lambda > 0,\ \lambda_1 > 0,\ 0 < p < \infty,\ 1 \le q < \infty,\ \omega(x) = x^\beta$ *and* $\omega^*(x) = x^{(1-\lambda)q+\beta}$. *Then the generalized Hardy-Littlewood-Polya operator* HL_λ *is bounded from* $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ *to* $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ *if and only if*

$$\lambda > \alpha - \lambda_1 + \frac{1}{q} + \frac{\beta}{q} > 0.$$

Then holds the inequality

$$\|HL_\lambda\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*) \longrightarrow M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} \le \frac{\lambda C(\alpha, \lambda_1, p)}{\left(\lambda_1 - \alpha + \lambda - \frac{1}{q} - \frac{\beta}{q}\right)\left(\alpha - \lambda_1 + \frac{1}{q} + \frac{\beta}{q}\right)},$$

where $C(\alpha, \lambda_1, p)$ *is defined in the statement of Theorem* 1.

Finally, to conclude the paper we consider the kernel $K(x,y) = \frac{1}{x}\lambda_E(x,y), E = \{(x,y)|y < x\}$ which defines the Hardy-type operator.

COROLLARY 2.6. *Let* $\alpha, \beta \in \mathbb{R},\ 0 < p < \infty,\ 1 \le q < \infty,\ \lambda_1 > 0,\ \omega(x) = x^\beta$ *and* $\omega^*(x) = x^\beta$. *Then the Hardy operator* H *is bounded from* $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ *to* $M\dot{K}^{\alpha,\lambda_1}_{p,q}(\mathbb{R}^+)$ *if and only if*

$$\lambda_1 - \alpha - \frac{1}{q} - \frac{\beta}{q} > 0.$$

Then holds the inequality

$$\|H\|_{M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega^*) \longrightarrow M\dot{K}^{\alpha,\lambda_1}_{p,q}(\omega)} \le \frac{C(\alpha, \lambda_1, p)}{\lambda_1 - \alpha + 1 - \frac{1}{q} - \frac{\beta}{q}},$$

where $C(\alpha, \lambda_1, p)$ *is defined in the statement of Theorem* 1.

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Tserendorj Batbold
Department of Mathematics
National University of Mongolia
Ulaanbaatar, Mongolia

tsbatbold@hotmail.com

Mario Krnić
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3, 10000 Zagreb, Croatia

mario.krnic@fer.hr

Amarsaikhan Amarbayar
National University of Mongolia
Ulaanbaatar, Mongolia

amarbayar.amarsaikhan@empathy.edu.mn