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Example of data envelopment analysis method based on partially ordered set theory

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Abstract: This paper presents results from applying Data Envelopment Analysis (DEA) based on Partially Ordered Set Theory to selected macroeconomic data from Mongolia involving multiple inputs and outputs. The aim is to demonstrate how hybrid models such as `zhdea_or`, `zhdea_dual`, and `gydea_dual` can be utilized in efficiency evaluation within a macroeconomic framework.

Key words: Input and output components, Efficiency value, Efficiency frontier

1. Introduction

Data Envelopment Analysis (DEA) is a widely used method to evaluate the efficiency of decision-making units (DMUs) based on multiple input and output variables. DEA is a non-parametric approach in operations research and economics for assessing the relative performance of units that perform similar tasks.

Since the introduction of the original Charnes-Cooper-Rhodes (CCR) model, a variety of DEA models have been developed, including the Banker-Charnes-Cooper (BCC), Färe-Grosskopf (FG), Charnes-Cooper-Wei-Huang (CCWH), and Seiford-Thrall (ST) models. These models are derived from different theoretical assumptions and reflect diverse characteristics of production possibility sets and efficiency frontiers, leading to various interpretations of technical and scale efficiency.

Despite the wide applicability of traditional DEA models, they may not address all practical decision-making challenges. Consequently, extended models such as stochastic DEA, fuzzy DEA, super-efficiency DEA, network DEA, and multi-stage DEA have been proposed.

A significant advancement in DEA methodology was the proposal by Ma Zhanxin of a DEA model based on Partially Ordered Set Theory. This approach, further developed by Mu Ren and others, enhances DMU analysis by introducing partial order relationships, leading to new methods for projecting DMUs and classifying their relationships.

These models, particularly `zhdea_or`, `zhdea_dual`, and `gydea_dual`, provide new insights into DMU interactions and support efficiency analysis under variable returns to scale through specialized projection techniques and partial order matrices [1, 2, 5, 6, 8, 9].

2. Basic definitions and theorems

Definition 1. Efficiency of a DMU In DEA, the efficiency of a DMU is defined as the ratio of the weighted sum of outputs to the weighted sum of inputs. A DMU is considered efficient if it operates on the efficient frontier, meaning no other DMU can produce more outputs with the same or fewer inputs. Mathematically, the efficiency score ranges between 0 and 1, where a score of 1 indicates that the DMU is efficient relative to others in the dataset.

Assume that the input-output data of DMU_i and DMU_j in the CCR, ST, and FG models are expressed as

$$X_i = (x_{1i}, x_{2i}, \cdots, x_{mi})^T, \qquad Y_i = (y_{1i}, y_{2i}, \cdots, y_{si})^T, \tag{2.1}$$

$$X_j = (x_{1j}, x_{2j}, \cdots, x_{mj})^T, \qquad Y_j = (y_{1j}, y_{2j}, \cdots, y_{sj})^T. \tag{2.2}$$

Assuming that each component of input and output is greater than zero, if there is a zero or negative input and output component, then all input and output data can be added with a positive number to ensure that all data are greater than zero. In this case, there is

$$a_p(p = 1, 2, \cdots, m), \qquad b_h(h = 1, 2, \cdots, s), \tag{2.3}$$

so that

$$x_{pi} = a_p x_{pj}, \;\; y_{hi} = b_h y_{hj}. \tag{2.4}$$

If

$$k_{ij} = min\{a_1, a_2, \cdots, a_m\}, \qquad r_{ij} = max\{b_1, b_2, \cdots, b_s\}, \tag{2.5}$$

then

$$x_{pi} \geq k_{ij} x_{pj}, \qquad y_{hi} \leq r_{ij} y_{hj}. \tag{2.6}$$

Definition 2. If $a_1 = a_2 = \cdots = a_m = b_1 = b_2 = \cdots = b_s$, then DMU_i and DMU_j in CCR, ST, and FG models are equivalent, denoted as $DMU_i \sim DMU_j$.

Definition 3. If $k_{ij} \geq r_{ij}$, then the DMU_i and DMU_j in the CCR, ST, and FG models are said to have an order relation «", which is denoted as $DMU_i < DMU_j$.

Definition 4. If $k_{ij} > r_{ij}$ and $k_{ji} > r_{ji}$, then DMU_i and DMU_j in CCR, ST, and FG models are said to have a strict order relation « $<$ ", which is denoted as $DMU_i << DMU_j$.

Definition 5. If $k_{ij} < r_{ij}$, then there is no order relation between DMU_i and DMU_j in CCR, ST, and FG models.

Definition 6. Let $(P, <)$ be a partially ordered set, $x \in P$, if for any $y \in P$ if $x < y$, then $y = x$, x is called a maximal element of the partially ordered set $(P, <)$.

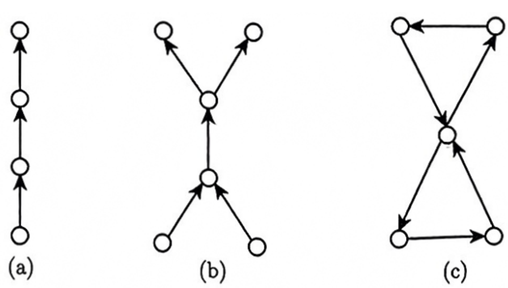

Figure 1: *Examples of totally ordered sets (a), partially ordered sets (b), and unordered sets (c).*

Theorem 1. In the CCR, ST, and FG models, the dominance relation "$<$" defined in Definition 3 constitutes a strict partial order on the set of DMUs.

Proof 1. We verify the three properties of a strict partial order:

- **Irreflexivity:** By definition, a DMU cannot dominate itself, i.e., $DMU_i < DMU_i$ never holds.
- **Antisymmetry:** Suppose $DMU_i < DMU_j$ and $DMU_j < DMU_i$. Then, for all p and h, there exist scalars $a_p, b_h > 0$ such that

$$x_{pi} = a_p x_{pj}, \quad y_{hi} = b_h y_{hj}, \tag{2.7}$$

  and both

$$k_{ij} = \min\left\{\frac{x_{pi}}{x_{pj}}\right\} \geq r_{ij} = \max\left\{\frac{y_{hj}}{y_{hi}}\right\}, \quad k_{ji} \geq r_{ji}. \tag{2.8}$$

  From this, we derive that all $a_p = 1$ and $b_h = 1$, implying $X_i = X_j$ and $Y_i = Y_j$, i.e., $DMU_i = DMU_j$.
- **Transitivity:** Suppose $DMU_i < DMU_j$ and $DMU_j < DMU_l$. Then there exist scalars such that

$$X_i = A \cdot X_j, \quad X_j = \bar{A} \cdot X_l \Rightarrow X_i = A \cdot \bar{A} \cdot X_l, \tag{2.9}$$

  and similarly for outputs. This leads to

$$k_{il} = k_{ij} \cdot k_{jl} \geq r_{ij} \cdot r_{jl} = r_{il}, \tag{2.10}$$

  hence $DMU_i < DMU_l$.

Thus, the "$<$" relation is a strict partial order. ■

Theorem 2. If DMU_i is efficient in the CCR, ST, or FG models, then DMU_i is a maximal element with respect to the "$<$" relation.

Proof 2. Assume DMU_i is efficient but not maximal. Then there exists $DMU_j \neq DMU_i$ such that $DMU_j < DMU_i$, i.e.,

$$x_{pi} \geq k_{ij} x_{pj}, \quad y_{hi} \leq r_{ij} y_{hj}, \quad \text{with } k_{ij} \geq r_{ij}. \tag{2.11}$$

Case 1: $k_{ij} > r_{ij}$. Then for all $u, v > 0$,

$$\frac{u^T Y_i}{v^T X_i} < \frac{u^T Y_j}{v^T X_j}, \tag{2.12}$$

contradicting the efficiency of DMU_i.
Case 2: $k_{ij} = r_{ij}$. Since $DMU_i \neq DMU_j$, there exists some p or h such that strict inequality holds, again implying DMU_i is less efficient than DMU_j. In both cases, DMU_i cannot be efficient, which contradicts the assumption. Hence, DMU_i must be maximal. ■

Theorem 3. If DMU_i is strictly dominated by DMU_j (i.e., $DMU_i << DMU_j$), then DMU_i is inefficient.

Proof 3. By definition, $DMU_i << DMU_j$ implies:

$$x_{pi} \geq k_{ij} x_{pj}, \quad y_{hi} < k_{ij} y_{hj}, \quad \text{with } k_{ij} > r_{ij}. \tag{2.13}$$

Thus, for all $u, v > 0$:

$$\frac{u^T Y_i}{v^T X_i} < \frac{u^T Y_j}{v^T X_j}, \tag{2.14}$$

so DMU_i is inefficient. Conversely, inefficiency does not imply strict domination by another DMU. ■

Theorem 4. If $DMU_i < DMU_j$ and $DMU_i \neq DMU_j$ in the CCR, ST, or FG models, then DMU_i is at most weakly efficient.

Proof 4. Given $DMU_i < DMU_j$, we have:

$$x_{pi} \geq k_{ij} x_{pj}, \quad y_{hi} \leq r_{ij} y_{hj}, \quad \text{with} \quad k_{ij} \geq r_{ij}. \tag{2.15}$$

If $k_{ij} > r_{ij}$: Then $DMU_i << DMU_j$, and by Theorem 3, DMU_i is inefficient.
If $k_{ij} = r_{ij}$: Since $DMU_i \neq DMU_j$, some strict inequality exists, leading to:

$$\frac{u^T Y_i}{v^T X_i} < \frac{u^T Y_j}{v^T X_j}, \quad \forall u, v > 0, \tag{2.16}$$

so DMU_i is not fully efficient. With zero-valued weights, it may appear efficient in certain weighted evaluations, but at best it is weakly efficient. ■

Theorem 5. In the CCR, ST, and FG models, if there is no dominance between DMU$_i$ and DMU$_j$, then there exist weight vectors $(u_1, v_1) \geq 0$ and $(u_2, v_2) \geq 0$ such that:

$$\frac{u_1^T Y_i}{v_1^T X_i} < \frac{u_1^T Y_j}{v_1^T X_j}, \quad \frac{u_2^T Y_j}{v_2^T X_j} < \frac{u_2^T Y_i}{v_2^T X_i}. \tag{2.17}$$

Proof 5. Since neither $DMU_i < DMU_j$ nor $DMU_j < DMU_i$ holds, we have:

$$k_{ij} < r_{ij}, \quad k_{ji} < r_{ji}. \tag{2.18}$$

Thus, for some input p and output h:

$$\frac{y_{hi}}{x_{pi}} < \frac{y_{hj}}{x_{pj}}, \quad \text{and} \quad \frac{y_{hj}}{x_{pj}} < \frac{y_{hi}}{x_{pi}}. \tag{2.19}$$

By assigning positive weights to the corresponding h and p, we obtain weight vectors (u_1, v_1) and (u_2, v_2) satisfying the inequalities. Hence, the efficiency ordering depends on the chosen weights. ■

Theorem 6. In the CCR, ST, and FG models, if DMU_i is a maximal element but not efficient, then for all $u, v > 0$ and for all j:

$$\frac{u^T Y_j}{v^T X_j} \leq \frac{u^T Y_i}{v^T X_i}. \tag{2.20}$$

Moreover, for any $u, v > 0$, there exists such a DMU_i satisfying:

$$\frac{u^T Y_i}{v^T X_i} \leq \frac{u^T Y_l}{v^T X_l}, \quad \forall l. \tag{2.21}$$

Proof 6. Assume there exists DMU_j and $u, v > 0$ such that:

$$\frac{u^T Y_j}{v^T X_j} > \frac{u^T Y_i}{v^T X_i}. \tag{2.22}$$

Then DMU_i cannot be the maximum under (u, v), contradicting its maximality. This implies DMU_i must be efficient under all such weights, contradicting the assumption. Thus, the inequality must hold for all j. ■

Theorem 7. In the CCR, ST, and FG models, if DMU_i is a maximal element and there exists $u, v > 0$ such that:

$$\frac{u^T Y_j}{v^T X_j} \leq \frac{u^T Y_i}{v^T X_i}, \quad \forall j, \tag{2.23}$$

then DMU_i is efficient.

Proof 7. Follows directly from Theorem 6. ■

3. Algorithm for Determining the Partial Order Relationship of Decision-Making Units

According to Theorem 3, in the CCR, ST, and FG models, the efficient decision-making units (DMUs) are the maximal elements in the corresponding partially ordered set. The algorithm for determining this relationship proceeds as follows:

- **Step 1.** Normalize the input-output data.
- **Step 2.** Select a DEA model (CCR, ST, or FG), and determine the (strict) partial order relationships between DMUs using Definitions 3 and 4.
- **Step 3.** Compute the efficiency score for each DMU.
- **Step 4.** Calculate the average input and output values for each DMU.
- **Step 5.** Plot a distribution diagram based on the efficiency scores.
- **Step 6.** Connect DMUs based on their partial order relationships.

Note: The partially ordered DEA method uses the algorithms `zhdea_or`, `zhdea_dual`, and `gydea_dual`, which integrate both the primal and dual forms of DEA. The general DEA model used is:

$$\theta \to \min,$$
$$ZH_D = \begin{cases} \sum\limits_{j=1}^{n} X_j \lambda_j \leq \theta X_0 \\ \sum\limits_{j=1}^{n} Y_j \lambda_j \geq Y_0 \\ \delta_1 \left(\sum\limits_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} \lambda_{n+1} \right) = \delta_1 \\ \lambda_j \geq 0, \quad j = 1, \ldots, n, n+1 \end{cases} \tag{3.1}$$

Model variants based on parameters:

- $\delta_1 = 0$: CCR model
- $\delta_1 = 1$, $\delta_2 = 0$: BCC model
- $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 0$: FG model
- $\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 1$: ST model

Slack-based extension (with ε adjustment):

$$[\theta - \varepsilon(e_i^T S^- + e_o^T S^+)] \to \min,$$
$$ZH_D^\varepsilon = \begin{cases} \sum\limits_{j=1}^{n} X_j \lambda_j + S^- = \theta X_0 \\ \sum\limits_{j=1}^{n} Y_j \lambda_j - S^+ = Y_0 \\ \delta_1 \left(\sum\limits_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} \lambda_{n+1} \right) = \delta_1 \\ S^- \geq 0, \quad S^+ \geq 0, \quad \lambda_j \geq 0, \quad j = 1, \ldots, n, n+1 \\ e_i = (1, \ldots, 1)^T \in \mathbb{R}^m, \quad e_o = (1, \ldots, 1)^T \in \mathbb{R}^s \end{cases} \tag{3.2}$$

Alternatively, if normalized data X_S and Y_S are used:

$$ZH_D^{\varepsilon} = \begin{cases} [\theta - \varepsilon(e_i^T S^- + e_o^T S^+)] \to \min, \\ \sum_{j=1}^{n} X_{Sj}\lambda_j + S^- = \theta X_{S0} \\ \sum_{j=1}^{n} Y_{Sj}\lambda_j - S^+ = Y_{S0} \\ \delta_1 \left(\sum_{j=1}^{n} \lambda_j + \delta_2(-1)^{\delta_3}\lambda_{n+1} \right) = \delta_1 \\ S^- \geq 0, \quad S^+ \geq 0, \quad \lambda_j \geq 0, \quad j = 1, \ldots, n, n+1 \\ e_i = (1, \ldots, 1)^T, \quad e_o = (1, \ldots, 1)^T \end{cases} \tag{3.3}$$

Notation:

- X, Y: Input-output data of DMUs
- θ: Efficiency score
- λ_j: Intensity variables
- S^-, S^+: Input and output slacks
- mx: DEA model used
- $compare\_dmu$: Partial order relationship matrix
- e_value: Efficiency values

4. Result

In this example, Mongolia's macroeconomic performance in 2023 is evaluated using DEA. Each Decision-Making Unit (DMU) has two input indicators and one output indicator. The input-output data are shown in the table below.

Table 1: Sample Data for Model

DMU	Divisions of economic activities	Employment	Investment (mln.tug)	GDP, at current prices (mln.tug)
1	Agriculture, forestry, finishing and hunting	293757	184006.8	6988223.0
2	Mining and quarrying	67219	13531894.1	19814080.3
3	Construction	79821	1225141.9	2269770.1
4	Financial and insurance activities	20381	692133.1	3258459.2
5	Education services	123865	671353.9	2678762.4
6	Human health and social welfare activities	52815	723673.5	1506947.1
7	Wholesale and retail trade	163682	2236606.6	7316253.2

The DEA analysis was conducted using hybrid algorithms based on Partially Ordered Set Theory: `zhdea_or`, `zhdea_dual`, and `gydea_dual`. The results include efficiency scores (`e_value`) and the reference matrix (`compare_dmu`):

```
compare_dmu =

         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0    2.0735         0    1.0051    1.1774
         0         0         0         0         0         0         0
    1.0383         0         0    1.1479         0         0         0
         0         0         0    1.5831         0         0    1.1167
         0         0         0    1.9862         0         0         0


e_value =

    1.0000    1.0000    0.3389    1.0000    0.4988    0.3697    0.5804
```

Figure 2: *Partial order relationships and efficiency values of DMUs*

Key insights:

- `e_value` shows the efficiency score of each DMU. For instance, DMUs 1, 2, and 4 are efficient (`e_value = 1.0000`), while DMU3 (`0.3389`) and DMU5 (`0.4988`) are inefficient.
- `compare_dmu` shows the efficient DMUs that each inefficient DMU is compared against. For example, DMU3 is referenced to DMU4, DMU6, and DMU7 with respective weights of 2.0735, 1.0051, and 1.1774.

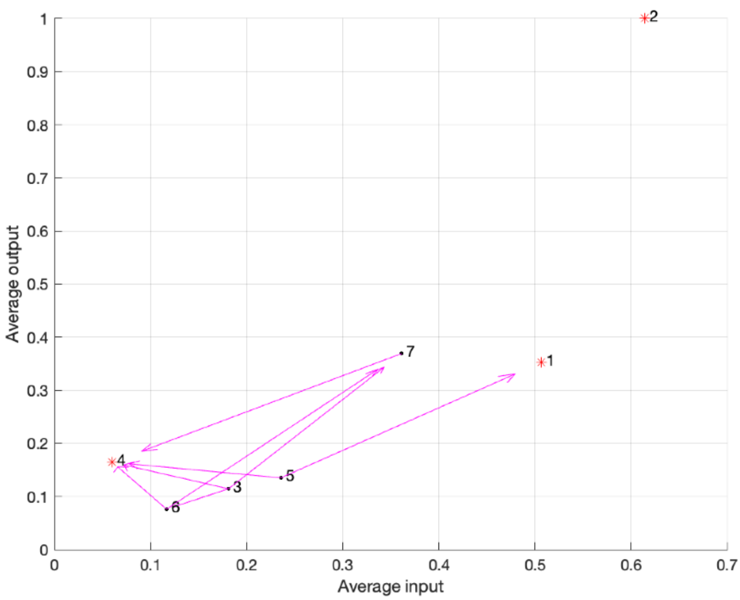

Figure 3: Partial order relationship diagram between DMUs

Diagram interpretation:

- **X-axis**: Average input; **Y-axis**: Average output
- **Red asterisks (*)**: Efficient DMUs on the frontier (e.g., DMU1, DMU2, DMU4)
- **Black dots with labels**: Inefficient DMUs
- **Magenta arrows**: Improvement paths toward efficient reference units

Observations:

- DMU2 demonstrates the best input-output ratio and is a key benchmark.
- DMUs 1 and 4 also form the efficiency frontier.
- DMUs 3, 5, 6, and 7 are inefficient and require adjustment toward efficient peers, as shown by the magenta arrows.

This analysis, based on partially ordered set theory, helps identify efficiency gaps and provides direction for improvement by benchmarking against high-performing units.

5. Discussion

As of this writing, there are no known published studies applying these specific models to macroeconomic analysis in Mongolia. While DEA has seen broad usage in economic evaluation, the application of Partially Ordered Set Theory-based DEA models in this context remains an unexplored area of research.

6. Conclusions

The combination of Data Envelopment Analysis (DEA) and Partially Ordered Set Theory proves to be an effective approach for evaluating the performance of macroeconomic sectors. This hybrid methodology not only enables a comparative assessment of relative efficiency among decision-making units but also provides a structured framework for identifying specific pathways toward improved performance. By illustrating how inefficient units can move toward efficiency frontiers through benchmarking against high-performing peers, the model offers valuable insights for policymakers and economists. Such insights can support evidence-based decision-making, inform sectoral development strategies, and enhance the overall effectiveness of macroeconomic planning.

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