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Graphs, contraction mappings and minors

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Abstract: This paper contains a review of topics on morphisms, contractions and minors, and several basic results that have been used repeatedly in the literature. Properties of contraction mappings and of cyclic connectivity are reviewed. The binary relation of minor inclusion is not far from being a partial order in any family of finite graphs. We also prove that if G is a 3-connected, internally 4-connected cubic graph that is not cyclically 4-connected then $G \simeq K_4$ or $G \simeq K_2 \Box K_3$. Diagrams as in other mathematical subjects such as algebra ought to be used for graph theory. Diagrams make concepts precise and clear, and avoid long verbal descriptions.

Key words: categories, contraction, cyclic connectivity, graphs, minors, morphisms, relational systems

1. Introduction

1.1. Graphs and Relations

Definition 1.1. Let V and E be sets with $V \neq \emptyset$. Then a mapping $G : E \to V \times V$ is called a *graph*.

This definition captures exactly the essence of the concept. The definition is adequate and convenient for generalizations and specializations. It must be noticed that the graph of a real-valued function of a real variable in analysis, is a special instance of Definition 1.1: the point $(x, f(x)) \in E$ of the Euclidean plane E is mapped to the ordered pair of real numbers $(x, f(x))$.

If G is an injective mapping, then since an injective mapping $G : E \to V \times V$ may be considered as a subset inclusion, we have the concept of a binary relation on V. Definition 1.1 is based only on well known concepts of sets and mappings. Hence the concept of a graph, in principle, does not present a new concept.

If V and E are finite sets then G is called *finite*; if E is irreflexive, that is, if for each $a \in V$, $(a, a) \notin E$, then G is *loopless*; if E is symmetric then the tuples in E may be considered subsets of cardinality 2 and hence G is *undirected* and *simple*; this is equivalent to an agreement: $ab = (a, b)$ and $ab = ba$; if E is antisymmetric then G is *oriented*; if E is antisymmetric and transitive then G is a *partial order*. Let V be a finite set and $r \geq 0$ be an integer. Denote

$$V^r := \underbrace{V \times \cdots \times V}_{r \text{ fold}},\ [V]^r := \binom{V}{r} := \{S \subseteq V : |S| = r\},\ D_V := \{(a, a) : a \in V\}.$$

Then the set D_V is called the *diagonal* of $\times$. Notice that the diagonal is also a binary relation on V. Let E be a binary relation on V. The *symmetric closure* $s(E)$ of E is

$$s(E) := E \cup \{(b, a) : (a, b) \in E\}.$$

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The natural projection

$$p_s : s\,[V \times V - D_V] \to [V]^2 := \binom{V}{2}$$

of the symmetric closure forgets the *order* of ordered tuples and maps ordered tuples to subsets of cardinality 2. If

$$G : E \to [V]^2 = \binom{V}{2}$$

is a bijection, then G is called the *complete* graph of order $|V| = n$ and is denoted $G = K_n$.

Basic concepts of graph theory such as subgraphs, induced subgraphs, spanning subgraphs, incidence, adjacency may all be defined in terms of diagrams of mappings. We will not expand on these in this note.

Consider a few simple generalizations.

Definition 1.2. Let V be a set and let $E = (E_1, \ldots, E_s)$ be a collection of sets. If

$$G = (G_1, \ldots, G_s)$$

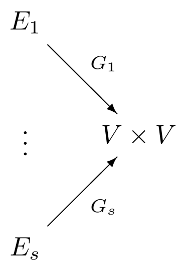
and

$$G_i : E_i \to V \times V,\ i = 1, \ldots, s$$

then

$$G : (E_1, \ldots, E_s) \to V \times V$$

is called a *graph system*. A graph system may be presented by the diagram



This seems to be an obvious generalization of the concept of a graph as given in Definition 1.1. The diagram reveals also that a graph system is a collection of graphs over the same set $V(G) = V$.

The second generalization is at the head of the arrow in the diagram of Definition 1.1.

Definition 1.3. Let V and E be sets.

$$G : E \to \underbrace{V \times \cdots \times V}_{m \text{ fold}}$$

is called an *m-ary relational system*. A *relational system* is

$$E \xrightarrow{G} \bigcup_{k=1}^{m} V^k$$

This is a proper generalization. Notice that this is more general than the concept of a set system or a hypergraph. Notice that at the head of the arrow is the union of cartesian products, instead of

$$\bigcup_{k=1}^{m} [V]^k := \{S \subseteq V : 1 \leq |S| \leq m\}.$$

For $R \subseteq V^m$, the *symmetric closure* or S_m-*closure* $s(R)$ is the quotient defined by the binary relation

$$(v_{p(1)}, v_{p(2)}, \ldots, v_{p(m)}) \sim (v_1, v_2, \ldots, v_m)$$

for all $p \in S_m$ the symmetric group on $\{1, \ldots, m\}$. That is, $s(R)$ is the quotient of the transitive action of the symmetric group S_m, then the usual set system (hypergraph) may be presented by the diagram

$$E \xrightarrow{\;G\;} \bigcup_{k=1}^{m} \binom{V}{k} \; .$$

A more general concept of a *relational system* may be obtained by a generalization on both head and tail of the arrow in Definition 1.1, and insisting that the mappings concerned are injective.

Definition 1.4. Let V be a set and let $E = (E_1, \ldots, E_s)$ be a collection of sets. If

$$G = (G_1, \ldots, G_s)$$

and

$$G_i : E_i \to \bigcup_{k=1}^{m} [V]^m, \; i = 1, \ldots, s$$

then

$$G : (E_1, \ldots, E_s) \to \bigcup_{k=1}^{m} [V]^m$$

is called a *relational system*. A relational system may be presented by the diagram

$$\begin{array}{ccc} E_1 & & \\ & \searrow^{G_1} & \\ \vdots & & \bigcup_{k=1}^{m} [V]^m \\ & \nearrow_{G_s} & \\ E_s & & \end{array}$$

If each G_i in Definition 1.4 is injective, then we have the definition of a simple relational system as in [10, page 28].

Definition 1.5. Let V be a set and let E be a collection of relations over V. Then $G = (V, E)$ is called a *simple relational system*. Let $k_1 < k_2 < \cdots < k_m$ and let the number of distinct k_i-ary relations in E be r_i. Then the symbol $(k_1^{r_1}, k_2^{r_2}, \ldots, k_m^{r_m})$ is called the *type* of G. The integer k_m is called the *arity* of E and hence of G.

2. Research materials and data

2.1. Categories of Graphs

There are many different categories of graphs (See [12–14]). For a formal and comprehensive treatment of categories, morphisms and functors, see [15]. In this section, based on an excerpt from [12, 14], some fundamental categories of graphs will be reviewed.

Definition 2.1. Let G and H be graphs and let $f : G \to H$ be a mapping. If for every $u, v \in V(G)$, $uv \in E(G) \Rightarrow f(u)f(v) \in H$ then f is called a *quasi homomorphism.* If for every $u, v \in V(G)$, $uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$ then f is called a *homomorphism.* An injective homomorphism is called a subgraph *inclusion* or *embedding.* A homomorphism $f : G \to G$ is called an *endomorphism* of G. A bijective homomorphism whose inverse mapping is also a homomorphism is called an *isomorphism.* A *comorphism* is a mapping $f : G \to H$ such that $f(x)f(y) \in E(H) \Rightarrow xy \in E(G)$.

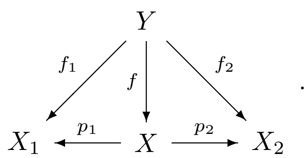
Definition 2.2 (Notations)**.** The category of all graphs with graph homomorphisms as morphisms is denoted by $\mathscr{G}$. The category of all graphs with quasi homomorphisms as morphisms is denoted by $\mathscr{QG}$. The category of all graphs with comorphisms as morphisms is denoted by $\mathscr{CG}$.

Compositions of morphisms satisfy the categorical axioms for compositions, in each of the three categories.

Products

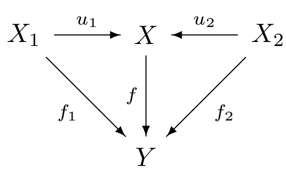
The binary operations of graphs typically include products. Natural products in graph categories are special types of products.

Consider the category $\mathscr{S}$ of sets and mappings. Let $X_1, X_2 \in \mathscr{S}$. A pair $(X, (p_1, p_2))$ with $p_1 : X \to X_1$, $p_2 : X \to X_2$ is called (the categorical) *product* of X_1, X_2 in $\mathscr{S}$ if (1) p_1, p_2 are morphisms in $\mathscr{S}$; and (2) $(X, (p_1, p_2))$ solves the universal problem: for every set Y and for every mappings $f_1 : Y \to X_1$, $f_2 : Y \to X_2$ there exists a unique mapping $f : Y \to X$ such that the following diagram commutes.



Theorem 2.1. *$(X_1 \times X_2, (p_1, p_2))$ is the product of X_1 and X_2 in $\mathscr{S}$.*

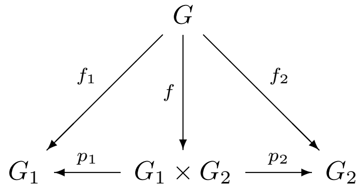
A pair $((u_1, u_2), X)$ is called the *coproduct* of X_1, X_2 in $\mathscr{S}$ if $u_1 : X_1 \to X$, $u_2 : X_2 \to X$ are mappings such that for every set Y and for every mappings $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ there exists exactly one mapping $f : X \to Y$ such that the diagram



is commutative. Notice that this diagram is obtained by a reversal of arrows in the diagram for product.

Theorem 2.2. *$((u_1, u_2), G_1 \cup G_2))$ is the coproduct of G_1 and G_2 in $\mathscr{G}$.*

The *categorical product* of graphs G_1 and G_2 may be defined by the requirement that for every graph G and homomorphisms $f_1 : G \to G_1$ and $f_2 : G \to G_2$, there exists a unique homomorphism $f : G \to G_1 \vee G_2$ so that the following diagram is commutative.



where p_1 and p_2 are natural projections (homomorphisms). It is in this sense that the product $\times$ ought to be called the *cartesian product*, or at least *categorical product*. This categorical product has been called *cross product* in graph theory literature. In addition to a wide range of confusion in naming, the product $\Box$ is called "*cartesian*" in various graph theory literature. This is a mistake. We shall prove in this section that the product $\Box$ is *not cartesian*.

We recall the definition of the graph product $\times$ from [10, page 37]. Let H and J be graphs. The product $H \Box J$ is defined to have

$$\begin{aligned} V(H \times J) &= V(H) \times V(J) \text{ cartesian product of sets} \\ E(H \times J) &= \{\{(u,v),(x,y)\} : ux \in E(H) \text{ and } vy \in E(J)\}. \end{aligned}$$

We recall the definition of the graph product $\Box$ [11, page 28]. Let H and J be graphs. The product $H \Box J$ is defined to have

$$\begin{aligned} V(H \Box J) &= V(H) \times V(J) \text{ cartesian product of sets} \\ E(H \Box J) &= \{\{(u,v),(x,y)\} : u = x, vy \in E(J) \text{ or } v = y, ux \in E(H)\}. \end{aligned}$$

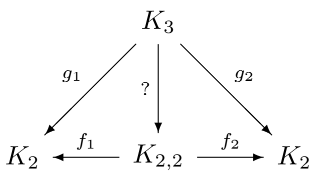
The *strong product* $\boxtimes$ is defined to be the union of the products $\Box$ and $\times$:

$$H \boxtimes J = (H \times J) \cup (H \Box J).$$

We show by an example that the graph product $\Box$ is *not cartesian*. If the graph product $\Box$ is cartesian then the morphisms must be contractions. Consider the graphs $K_{2,2}$ and K_3. There exist contractions

$$f_1, f_2 : K_{2,2} \to K_2 \text{ and } g_1, g_2 : K_3 \to K_2.$$

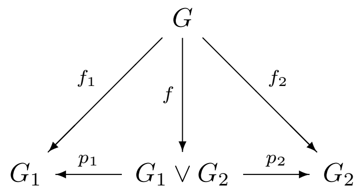
We have $K_3 \not\simeq K_{2,2}$. Hence the following diagram may not be completed by a unique isomorphism.



Although f_1, f_2 and g_1, g_2 are contractions (morphisms), there exists no isomorphism $K_3 \to K_{2,2}$. Hence there exists no isomorphism to complete the diagram in the category of finite undirected graphs with contractions as morphisms. Therefore, $\Box$ is not a natural product in the category of finite simple undirected graphs with contractions as morphisms.

Since the graph product $\times$ is natural in the category of finite simple undirected graphs with homomorphisms as morphisms, the product $\times$ ought to be called *cartesian* and a suggestive name for the product $\Box$ is *direct*.

The *disjunction* of graphs G_1 and G_2 may be defined by the requirement that for every graph G and homomorphisms $f_1 : G \to G_1$ and $f_2 : G \to G_2$, there exists a unique homomorphism $f : G \to G_1 \vee G_2$ so that the following diagram is commutative.



where p_1 and p_2 are natural projections.

Binary operations of graphs are interpreted categorically in [12, 14], where the following were among results established there.

1. The product $G_1 \times G_2$ with projections is a product of G_1 and G_2 in $\mathscr{G}$.
2. The product $G_1 \boxtimes G_2$ with projections is a product of G_1 and G_2 in $\mathscr{Q}\mathscr{G}$.
3. The disjunction with projections is a product of G_1 and G_2 in $\mathscr{C}\mathscr{G}$.

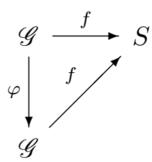
These capture the essence of the products concerned in the respective categories.

Categories $\mathscr{G}$, $\mathscr{C}\mathscr{G}$ and $\mathscr{Q}$ also have coproducts and tensor products [12, 14]. It was also shown in [12, 14] that products and coproducts in these three categories have right adjoints.

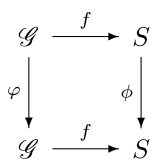
2.1.1. Graph invariants

Investigations about invariants in various fields of mathematics concern the action of a group, usually a subgroup of the automorphism group. Combinatorics is not an exception.

Definition 2.3. Let $\mathscr{G}$ be a set of graphs and S be a set with an equivalence relation $\simeq$. A mapping $f : \mathscr{G} \to S$ is called an *invariant* of graphs if for every $G, H \in \mathscr{G}$, $G \simeq H \Rightarrow f(G) \simeq f(H)$. A graph invariant f may be presented by the following diagram.



With the condition $G \simeq H \Rightarrow f(G) \simeq f(H)$, a graph invariant for a category of graphs may be presented by the diagram



where φ is a graph isomorphism and ϕ is an isomorphism of S.

Of course, for S a set of numbers or a set of sequences of numbers, the second $\simeq$ is just $=$. In terms of mappings, a function taking its argument as a graph G is an invariant if for each automorphism φ of G, $f(\varphi(G)) = f(G)$, or simply, $f\varphi = f$, since the condition is assumed for every graph in the given category.

Examples of graph invariants include the degree sequence, the automorphism group, spectra, the determinant of the adjacency matrix. Any integer valued invariant is an example of a graph parameter. These include, of course, the order, size, diameter, girth, circumference, connectivity, edge connectivity, independence number, covering number, chromatic number, edge chromatic number, and matching number; to name a few. A significant portion of graph theory is the study of graph parameters.

Definition 2.4. Let S be a partially ordered set with a maximum elelement and a minimum element. An invariant $f : \mathscr{G} \to S$ is called a graph *parameter*. A parameter f is said to satisfy the *intermediate value property*, if for each $s \in S$ satisfying $f_{\text{min}} \le s \le f_{\text{max}}$, there exists a graph $G \in \mathscr{G}$ with $|G| = n$ and $f(G) = s$. A parameter f is said to satisfy the *absolute intermediate value property*, if for each $s \in S$ satisfying $f_{\text{min}} \le s \le f_{\text{max}}$, there exists a graph $G \in \mathscr{G}$ with $f(G) = s$.

If a graph parameter $f \in \mathbb{Z}$, then the ring is understood to be embedded in an ordered field such as $\mathbb{Q}$. No problem was caused in the absence of this understanding. Examples of general problems on intermediate value theorems for graphs were proposed in [7, pages 59-60].

3. Methodology

3.1. Contractions and Minors

It is insisted throughout this note that only the well known concepts of sets and mappings are employed. This is to avoid long verbal descriptions. A definition of a concept is preferred to a description. It is seen in this section that the diagrams of arrows are powerful mathematical tools worthy of adoption.

Definition 3.1. Let $G = (V, E)$ be a graph. For $X, Y \subseteq V(G)$, denote

$$(X, Y) = \{xy : x \in X, y \in Y, xy \in E(G)\}.$$

A *contraction* of G is defined to be a partition $\{V_1, V_2, \ldots, V_s\}$ of V such that for each $i = 1, 2, \ldots, s$, the induced subgraph $G|_{V_i}$ is connected. This partition gives rise to a natural mapping from G to a graph H. The contraction (graph) H is the graph with

$$V(H) = \{V_1, V_2, \ldots, V_s\}, \quad E(H) = \{V_iV_j : i \ne j,\ (V_i, V_j) \ne \emptyset\}.$$

The mapping $f : G \longrightarrow H$ is called a *contraction* (mapping) from G onto H, and G is said to be *contractible* to H. H is called a *contraction* (graph) of G.

For an undirected graph G, the notation $[X, Y]$ is used. If the mention of a graph is absent and if $X \cap Y \ne \emptyset$ then $[X, Y] = K_{m,n}$ with $|X| = m$ and $|Y| = n$.

The graph K_1 is a contraction of any connected graph G since $\{V\}$ is a partition of V and $G = G|_V$ is connected. Any automorphism of G is a contraction since it is a permutation of the trivial partition of V into single vertices. In particular, $1 : G \to G$ is a contraction.

Suppose that $J \subseteq G$ is a connected subgraph. Then the contraction of J in G, denoted G/J, is given by the partition

$$\{V(J), \{v_1\}, \ldots, \{v_m\}\}$$

where $V(G) \setminus V(J) = \{v_i : 1 \le i \le m\}$. This special case covers the contraction of a single edge e with $J = K_2$.

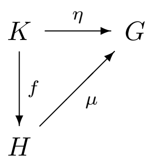
Definition 3.2. Let $r \ge 1$ be an integer and denote by $e^r = (uv)^r$ the presence of r parallel edges between vertices u and v in a multigraph. A contraction $f : G \to H$ is called a *faithful* contraction if

$$E(H) = \{(V_iV_j)^r : |(V_i, V_j)| = r, i \ne j\}.$$

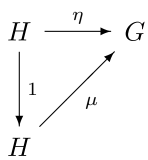
For an undirected graph, the notation $[V_i, V_j]$ may be abbreviated as V_iV_j. If we do not agree $UV = VU$ then for graphs, not necessarily undirected, the notation (U, V) may be denoted as UV. Both cases do not exclude the possibility that $U \cap V \neq \emptyset$.

Note that Definition 3.1 is adequate for generalizations to directed graphs, infinite graphs and set systems.

Definition 3.3. A graph H is a *minor* of G, if G has a subgraph contractible to H. That is, there is a subgraph $K \subseteq G$ and a contraction $f : K \to H$. This is equivalent to the statement that the following diagram is commutative.

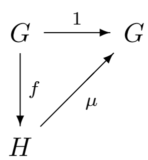


Notice that each of the three arrows are prefixed by the existential quantifier. The commutative diagram in this definition is adequate and versatile for proving basic properties of minor inclusions. First, since $1 : H \to H$ is a contraction, we have



Hence if $H \subseteq G$ then $H \leq G$. In words, every subgraph is a minor. An extreme special case of this statement is $G \leq G$. This is the reflexivity of the binary relation $\leq$.

If $f : G \to H$ is a contraction, then



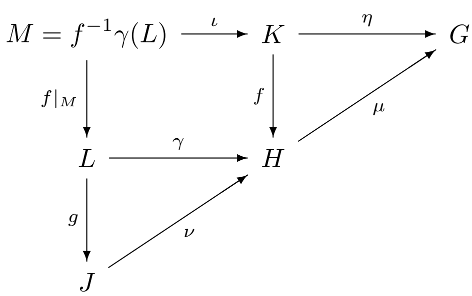
Hence, if $f : G \to H$ is a contraction, then $H \leq G$. In words, every contraction graph is a minor. Note that the converse is not true. For an example, $K_{3,3} \leq P$ where P is the Petersen graph, but there is no contraction $f : P \to K_{3,3}$.

The two simple examples of application of the arrow diagram proves already that minor inclusion is a common generalization of both the concept of a subgraph and the concept of a contraction graph.

The transitivity of the binary relation $\leq$ may also be established by using a diagram.

Proposition 3.1. *Let G, H, J be graphs. If $J \leq H$ and $H \leq G$ then $J \leq G$.*

Proof. Consider the diagram



In this diagram, $M = f^{-1}\gamma(L) \subseteq K \subseteq G$ is a subgraph of G, $f\iota = \gamma f|_M$, $f|_M(M) = f|_M f^{-1}\gamma(L) = \gamma(L) \simeq L$ and $gf|_M : M \to J$ is a contraction since composition of contractions is a contraction. By Definition 3.3, $J \leq G$.

4. The results

4.1. Orders in Graph Families

Definition 4.1. Let S be a set and $\leq$ be a binary relation on S. If $\leq$ is reflexive and transitive then it is called a *quasi order* on S. If $\leq$ is antisymmetric and transitive then it is called a *partial order* on S.

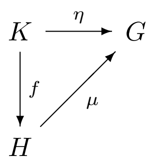
Proposition 4.1. *Let S be a family of finite simple undirected graphs. Then the binary relation $\leq$ of minor inclusion is a quasi order on S.*

Proof. It was verified in the preceding section that $\leq$ is reflexive and transitive. By Definition 4.1, $\leq$ is quasi order.

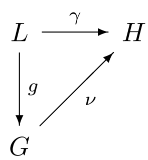
The binary relation of minor inclusion is not far from being antisymmetric for any family of finite simple undirected graphs.

Proposition 4.2. *Let G and H be finite graphs. If $H \leq G$ and $G \leq H$ then $G \simeq H$.*

Proof. Suppose that G and H are finite graphs and that $H \leq G$ and $G \leq H$. Then we have diagrams



and



Since $|G| \leq |L| \leq |H|$ and $|H| \leq |K| \leq |G|$, we have $|G| = |H|$. Hence G is a spanning subgraphs of H and H is a spanning subgraph of G. In particular, $V(G) = V(H)$. By definition, $E(H) \subseteq E(G)$ and $E(G) \subseteq E(H)$. Hence, $E(G) = E(H)$. Hence,

$$f : G \to H, \; g : H \to G$$

are both contractions and bijections. Since f and g are contractions, we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Hence $G \simeq H$.

Theorem 4.1. *Let S be a set of finite graphs and let $\leq$ be the binary relation of minor inclusion. Let $B \subseteq S$ be the set of minimal elements of $(S, \leq)$. If $\leq$ satisfies the Jordan-Dedekind descending chain condition then $|B|$ is finite.*

Proof. Let S be a set of finite graphs ordered under the binary relation $\leq$ of minor inclusion and let $B \subseteq S$ be the set of minimal elements of $(S, \leq)$. Since $\leq$ satisfies the Jordan-Dedekind descending chain condition, there is exists $H \in B$ and a strictly descending finite chain from G to H. Since $|G|$ is finite, $|B|$ is finite.

Note in passing that the principle of mathematical induction is valid on a set S if and only if the set S has a quasi ordering $\leq$ which satisfies the Jordan-Dedekind descending chain condition. Note that the condition that every antichain is finite is *not* necessary for the principle of mathematical induction to be applicable in S. A construction procedure based on the principle, however, relies on the finite antichain condition.

5. Discussion

5.1. Connectivity and Variations

In the study of connectivity of graphs, the theorem of Menger from 1927 occupy a central position.

Let G be a graph and $A, B \subseteq V(G)$. Denote by $f_G(A, B)$ the maximum number of (A, B)-paths in G, and by $s_G(A, B)$ the cardinality of a minimum separator that separates A from B in G.

As in [18, pp. 11-12] or [9, page 62], the following argument provides a clear indication to the procedure of reductions of graphs with a certain property that gives specific minor inclusions.

If $S \subset V(G)$ separates A from B in G, then every path from A to B intersects S. Hence $f_G(A, B) \leq s_G(A, B)$.

For $f_G(A, B) \geq s_G(A, B)$, apply induction on $\|G\|$. If $E(G) = \emptyset$ then

$$f_G(A, B) = s_G(A, B) = |A \cap B|.$$

Assume that there is an edge $uv \in E(G)$ and that for each H with $\|H\| < \|G\|$ the assertion of the theorem is true. Let $s = s_G(A, B)$. We prove that there are s internally disjoint paths from A to B in G. Consider $H = G/uv$. Let T be a minimum separator of H separating A/uv from B/uv. If $|T| \geq s$, then by the inductive hypothesis, $f_H(A/uv, B/uv) \geq s$. Thus, $f_G(A, B) \geq s$. Hence assume that $|T| < s$. Since S separates A from B, hence $|S| \geq s$. Hence $u, v \in S$ and $|S| = s$.

Consider $s_G(A, S)$. Since $s_G(A, B) = s$, $s_G(A, S) \geq s$. In fact, $s_{G-uv}(A, S) \geq s$ (the edge uv does not affect connectivity from A to S). Note also that these paths from A to S has distinct ends in S since $|S| = s$. Similarly, there exist s disjoint paths from S to B in $G - uv$. Then $f_G(A, B) \geq s$. This proves

Theorem 5.1 (Menger, 1927)**.** *Let G be a graph and $A, B \subseteq V(G)$. Then $f_G(A, B) = s_G(A, B)$.*

This proof of Menger's theorem, Thomassen's proof of Tutte's theorem on 3-connected graphs (see [9, page 45]) and Thomassen's proof of Kuratowski's theorem in [20] gave a clear evidence to the importance of contraction mappings and minors in graphs.

Indeed the proof above establishes

Theorem 5.2. *For any graph G, and any $e \in E(G)$, either*

$$f_{G/e}(A/e, B/e) = s_{G/e}(A/e, B/e) \Rightarrow f_G(A, B) = s_G(A, B)$$

or

$$f_{G-e}(A, B) = s_{G-e}(A, B) \Rightarrow f_G(A, B) = s_G(A, B).$$

Corollary 5.1 (Dirac, 1961)**.** *Let G be a k-connected graph. Then for every $A \subseteq V(G)$ with $|A| \leq k$, G has a circuit containing A.*

We omit the simple proof of this result of Dirac using Menger's theorem. This corollary motivated much research since 1961.

Definition 5.1. Let G be a connected graph and let $X \subseteq V(G)$ (resp. $S \subseteq E(G)$). If $G - X$ (resp. $G - S$) is not connected and each component of $G - X$ (resp. $G - S$) is cyclic, then X (resp. S) is called a *cyclic separator* (resp. *cyclic separating set* or *cyclic cut set* as used by many authors). The cardinality (resp. size) of a smallest cyclic separator (resp. cyclic separating set) is called the *cyclic connectivity* (resp. *cyclic edge connectivity*) of G. The cyclic (resp. edge) connectivity of G is denoted by $\kappa'(G)$ (resp. $\lambda'(G)$).

The concept of cyclic connectivity has been crucial in many results that have been obtained for cycles in cubic graphs [1–6]. We also note that the concept of feedback, or feedback number is also a variation of the concept of connectivity as note in [8, Section 7].

6. Conclusions

6.1. Properties of Contractions

We now turn to some basic properties of contractions and of cyclic connectivity. Included in this note for completeness, the proof of the following theorem is absolutely ground level, for it made reference only to definitions. Never appeared in this complete form, these results have been cited and applied in [1–6].

Theorem 6.1. *The following are equivalent.*

(1) *There is a contraction* $f : G \to H$*;*

(2) *There exists* $S \subseteq E(G)$ *such that for every* $e \in E(G)\backslash S$, $\omega(G|_S \cup e) < \omega(G|_S)$ *and for the set* $V(H)$ *of components of* $G|_S$ *there is a surjection* $\phi : E(G)\backslash S \to E(H)$*;*

(3) *There is a sequence*

$$G = G_0 < G_1 < \cdots < G_s = H$$

of graphs such that for each i, $0 \leq i \leq s-1$, *there is* $e_i \in E(G_i)$ *such that* $G_{i+1} = G_i/e_i$*;*

(4) *There is a surjective mapping* $f|_V : V(G) \to V(H)$ *such that for each* $v \in V(H)$, $G|_{f^{-1}(v)}$ *is connected and if* $X = f^{-1}(E(H)) \subseteq E(G)$ *then* $f|_X : X \to E(H)$ *is a bijection and* f *commutes with the incidence functions, that is,* $f\iota_G = \iota_H f$*;*

(5) *There is a spanning subgraph* $R \subseteq G$ *with components* $G_1, \ldots, G_s$ *such that* $H = G/G_1/G_2/\cdots/G_s$*;*

(6) *There is an quasi homomorphism* $f : G \to H$ *such that for each* $v \in V(H)$, $G|_{f^{-1}(v)}$ *is connected;*

(7) *There is an equivalence relation* $\sim$ *on* $V(G)$ *such that each equivalence class induces a connected subgraph of* G*.*

Proof.

(1)⇔(2):

Let $f : G \to H$ be a contraction mapping. By Definition 3.1, there exists a partition $\{V_1, V_2, \ldots, V_2\}$ such that for each $i \in \{1, 2, \ldots, s\}$, $G|_{V_i}$ is connected. Let $S = f^{-1}(E(H))$. Since $G|_{V_i}$ is connected, for every $e \in E(H) \setminus S$ we the number of components of $G|_{S\cup\{e\}}$ is at least one less than that of G_S. Since f is surjective, the mapping $\phi : E(G) \setminus S \to E(H)$ induced by f is surjective. Conversely, let $G_1, G_2, \ldots, G_s$ be the components of $G - S$ and let $V_i = V(G_i)$. Then the partition $\{V_1, V_2, \ldots, V_s\}$ determines a contraction mapping $f : G \to H$ with $V(H) = \{V_1, V_2, \ldots, V_s\}$.

(2)⇔(3):

Suppose that (2) is true. If $S = E(G)$, then $H = G$ and (3) is true. Let $e \in E(G) \setminus S$. Then G/e is a contraction of G with $G/e < G$. Conclusion (3) follows by mathematical induction. The composition of a finite sequence of contraction mappings is a contraction mapping. Hence (3) ⇒ (2).

(1)⇔(4):

Let $f : G \to H$ be a contraction mapping. By Definition 3.1, there exists a partition $\{V_1, V_2, \ldots, V_2\}$ such that for each $i \in \{1, 2, \ldots, s\}$, $G|_{V_i}$ is connected. Let $X = f^{-1}(E(H))$. Then $X \subseteq E(G)$, $f|_X$ is a surjective mapping and by Definition 3.1, $E(H) = \{V_iV_j : i \neq$

$j,\ [V_i, V_j] \neq \emptyset\}$. Hence $\iota_H f = f\iota_G$. Conversely, if $f|V$ is as in (4), then by Definition 3.1, f defines a contraction mapping $f : G \to H$.

(1)⇔(5):

Let $f : G \to H$ be a contraction mapping determined by a partition $\{V_1, V_2, \ldots, V_s\}$. Let $G_i = G|_{V_i}$. By Definition 3.1, for each $i \in \{1, 2, \ldots, s\}$, G_i is connected. Then $H = G/G_1/G_2/\cdots/G_s$. Notice that the order of contractions of the subgraphs is not material in the sequence. Conversely, if $R \subseteq G$ is a spanning subgraph with components $G_1, G_2, \ldots, G_s$, then let $V_i = V(G_i)$. Then $\{V_1, V_2, \ldots, V_s\}$ is a partition of $V(G)$ such that for each $i \in \{1, 2, \ldots, s\}$, $G|_{V_i}$ is connected. Definition 3.1, $f : G \to H$ is a contraction mapping.

(1)⇔(6):

By Definitions 2.1 and 3.1, a quasi homomorphism $f : G \to H$ such that for each $v \in V(H)$, $G|_{f^{-1}(v)}$ is connected is precisely a contraction mapping $f : G \to H$.

(1)⇔(7):

The set $\{V_1, V_2, \ldots, V_2\}$ is a partition of a set V if and only if there exists an equivalence relation $\sim$ on V with $V_1, V_2, \ldots, V_s$ as equivalence classes. By Definition 3.1, the partition determines a contraction mapping if and only if for each $i \in \{1, 2, \ldots, s\}$ $G|_{V_i}$ is connected.

Denote by $L(G)$ the line graph of G. A k-connected graph is k^+-*connected* if it has no nontrivial separator of cardinality k. Note that a 3^+-connected graph was called a quasi 4-connected graph in [17]. The concept parallel to this for edges was introduced in [16].

Theorem 6.2. *Let G be a cubic graph. Then the following are equivalent.*

(1) *G is cyclically 4-connected;*
(2) *G is 3^+-connected;*
(3) *$L(G)$ is 4-connected;*
(4) *If $f : G \to H$ is a faithful contraction and H is cubic then $f \in \mathrm{Aut}G$ and $H \simeq G$.*

With this statement, we only use the term given in (1), namely *cyclic connectivity*.

By Theorem 6.2 (4), all 3-connected cubic graphs may be constructed from cyclically 4-connected cubic graphs by lifting appropriate faithful contractions. This is a brief summary of [16, Theorem 9, page 393].

The concept of internally 4-connected graphs was given in [19].

Definition 6.1. Let G be a finite simple undirected graph. If $G_1, G_2 \subseteq G$, $G_1 \cup G_2 = G$ and $E(G_1) \cap E(G_2) = \emptyset$, then (F_1, G_2) is called a *separation* of G. The integer $|V(G_1) \cap V(G_2)|$ is called the order of the separation $\{G_1, G_2\}$.

If G is 3-connected and for every separation $\{G_1, G_2\}$ of G of order 3, either $|E(G_1)| \leq 3$ or $|E(G_2)| \leq 3$ G is called *internally* 4-connected.

Theorem 6.3. *Let G be a 3-connected, internally 4-connected cubic graph. If G is not cyclically 4-connected, then $G \simeq K_4$ or $G \simeq K_2 \Box K_3$.*

Proof. Suppose that G is an internally 4-connected cubic graph. Let $S \subseteq E(G)$ be a separating set with $|S| \leq 3$. Since G is 3-connected, $|S| \geq 3$ and hence $|S| = 3$. Suppose that the two components of $G - S$ are L and R. Denote by L_S and R_S the end vertices of S in L and in R respectively. If $|L| \geq 4$, then let

$$G_1 = L,\ G_2 = G|_{R \cup L_S}.$$

Then $\{G_1, G_2\}$ is a separation of order 3 such that both $\|G_1\|, \|G_2\| \geq 4$. This contradicts the assumption that G is internally 4-connected. Hence both $\|L\|, \|R\| \leq 3$. Therefore, either $G \simeq K_4$ or $G \simeq K_2 \Box K_3$.

By Theorem 6.3, the concept of internally 4-connected cubic graphs is a trivial concept in the foreground of Definition 5.1.

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