The Connection Between Pareto Optimality and Portfolio Growth Rate

Enkhbat Rentsen, Gantigmaa Ganlkhagva*

Institute of Mathematics and Digital Technology, Mongolian Academy of Sciences, Ulaanbaatar 13330, Mongolia

**Corresponding author: gantigmaa_g@mas.ac.mn; ORCID:0000-0002-9604-9233*

Article Info: Received: 2022.09.15; Accepted: 2022.10.01; Published: 2022.12.26

Abstract: Portfolio optimization plays an important role in investment sciences. We examine the classical Markowitz model from a viewpoint of Pareto optimality. We consider a multi-objective optimization problem by maximizing the return of a portfolio and minimizing risk. We show that for appropriate weights, the Pareto optimal solution of the multi-objective optimization is a solution to the problem of maximizing a portfolio growth rate. Numerical results were provided using Mathlab.

Key words: Markowitz Theory, Multi-Objective Efficiency

1. Introduction

In 1990, Mongolia transitioned to a market economy, but the stock market is still in its early stages of development, and people don't always participate in trading. However, since 2020, cryptocurrency has been traded, and people have become more interested in various financial assets and investments. Therefore it is important for individuals and firms to understand how to build a portfolio and how to maximize returns with minimal risk. Especially, understanding the optimal portfolio from a mathematical point of view is crucial. Markowitz's theory is based on the fact that the risk of an investment portfolio consisting of various financial instruments is lower than the investment risk of individual financial instruments. The Markowitz problem provides the foundation for the single-period investment theory. The problem explicitly addresses the trade-off between the expected rate of return and the variance of the rate of return in a portfolio [1]. In multi-objective optimization, when there are functions with opposing objectives, the optimal solution is called Pareto optimization. Pareto-optimal solutions can be said to be multi-objective optimal solutions because it is impossible to find an optimal solution without degrading one of the objectives. Therefore, Pareto optimization and Markowitz theory together determine the optimal investment portfolio. For our numerical experiment, taking into account the current economic situation, we selected assets with increasing returns and calculated the optimal investment portfolios in two ways. The first problem defines the optimal portfolio when shorting is possible, while the second problem finds the optimal portfolio when investments have only longed. Shorting is when an investor first borrows an asset, then sells it when the price is high, and then buys it back when the price drops to repay the loan. Investors use this strategy when they expect the price of the asset to fall. On the contrary, when an investor predicts that the price of an asset will rise, he/she buys the asset and sells it when the price increases, which is called longing.

2. Methodology

Assume that there are n assets. The expected rates of return are $r_1, r_2, \ldots, r_n$ and the covariance are $\sigma_{ij},\ i,j = 1,2,\ldots,n$. A portfolio is defined by a set of n weights $x_i, i = 1,2,\ldots,n$ such that $\sum_{i=1}^{n} x_i = 1$. Denote by $\bar{r}$ is the expected value of portfolio return, $\sum_{i=1}^{n} x_i r_i = \bar{r}$.
The Markowitz model is to minimize the variance of the portfolio [2]:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j \tag{2.1}$$

$$\sum_{i=1}^{n} x_i r_i = \bar{r} \tag{2.2}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{2.3}$$

Problem (2.1)-(2.3) is known as the convex minimization problem and is solved by the Lagrange method [3]. Introduce the functions f_1 and f_2 in the following:

$$f_1 = -\sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j,$$

$$f_2 = \sum_{i=1}^{n} x_i r_i,$$

$$S = \left\{ x \in R^n \middle| \sum_{i=1}^{n} x_i = 1,\ \sum_{i=1}^{n} x_i r_i = \bar{r} \right\}.$$

Now we consider the problem of minimizing variance and maximizing the return of the portfolio. This problem is formulated as a multi-objective optimization as follows:

$$\max f_1 = -\sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j \tag{2.4}$$

$$\max f_2 = \sum_{i=1}^{n} x_i r_i \tag{2.5}$$

Recall the definition of Pareto optimal solutions for problems (2.3) and (2.4):

Definition 2.1. [4] $\bar{x} \in S$ is called a Pareto optimal point of the problem (2.4)-(2.5) if there is no $\in S$ with

$$f_i(y) \geq f_i(\bar{y}),\ i = 1,2,$$
$$f(y) \neq f(\bar{y}),\ f = (f_1, f_2).$$

The Pareto optimal concept is the main optimality notion used in multi-objective optimization. The main approach for finding Pareto optimal solutions is the weighted sum approach. Therefore, one can introduce positive weights $\alpha_1 > 0,\ \alpha_2 > 0$, so we can formulate the corresponding scalarized optimization problem for problems (2.4)-(2.5).

$$\max_{x \in S} F(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) \tag{2.6}$$

A relationship between Pareto optimal solutions and solutions of the scalarized problem can be given by the following assertion.

Proposition 2.1. *A solution x^* to problem (2.6) is a Pareto optimal solution to the problem (2.4)-(2.5).*

Proof. Let x^* be a solution to the problem (2.6), but is not a Pareto optimal solution of problem (2.4)-(2.5). This means that there exists a point $\widetilde{x} \in S$ such that

$$f_i(\widetilde{x}) \geq f_i(x^*),\ i = 1,2\ with\ f_j(\widetilde{x}) > f_j(x^*),\ \exists j \in \{1,2\},$$
$$F(x^*) = \sum_{i=1}^{2} \alpha_i f_i(x^*) \leq \sum_{i=1}^{2} \alpha_i f_i(\widetilde{x}) < \sum_{i \neq j}^{2} \alpha_i f_i(\widetilde{x}) + f_j(\widetilde{x}) = F(\widetilde{x}).$$

This contradicts that x^* is a solution to the problem (2.6).
On the other hand for $\alpha_1 = \frac{1}{2},\ \alpha_2 = 1$. It has been shown that F is defined as

$$F = \sum_{j=1}^{n} x_j r_j - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j$$

is the growth rate of the portfolio [1]. Now we have the following two problems depending on a short and long position. □

Problem 1 (selling with Short):

$$\max F = \sum_{j=1}^{n} x_j r_j - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j,$$
$$\sum_{i=1}^{n} x_i = 1.$$

Problem 2 (selling long):

$$\max F = \sum_{j=1}^{n} x_j r_j - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j,$$
$$\sum_{i=1}^{n} x_i = 1,$$
$$x_j \geq 0,\ j = 1, 2, \ldots, n.$$

Introduce matrix notations for these problems:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\quad r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ n_n \end{pmatrix},\quad C = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nn} \end{pmatrix}.$$

Problem 1 can be written as follows:

$$\max_{x \in S} F = \langle r, x \rangle - \frac{1}{2} \langle Cx, x \rangle$$

where denotes the inner scalar product in R^n. $C > 0$ is a positive definite symmetric matrix. In order to solve problem 1, we write down the Lagrange function.

$$L(x, \lambda) = \langle r, x \rangle - \frac{1}{2} \langle Cx, x \rangle + \lambda_1 \left(\sum_{i=1}^{n} x_i - 1 \right) + \lambda_2 \left(\sum_{i=1}^{n} r_i x_i - r \right)$$

Optimality conditions are

$$\frac{dL}{dx} = r - Cx + \lambda_1 E + \lambda_2 r = 0,$$
$$Cx = r + \lambda_1 E + \lambda_2 r,\ x = C^{-1}\left(r + \lambda_1 E + \lambda_2 r\right).$$

By analogy, Problem 2 can be written as:

$$\begin{array}{l} \max F = \langle r, x\rangle - \frac{1}{2}\langle Cx, x\rangle, \\ \langle r, x\rangle = \bar{r}, \\ \sum_{i=1}^{n} x_i = 1, \\ x_j \geq 0,\ j = 1, 2, \dots, n. \end{array} \tag{2.7}$$

Problem 2 in matrix form is

$$\max F = \langle r, x\rangle - \frac{1}{2}\langle Cx, x\rangle$$

where, $S = \{x \in R^n | \sum_{i=1}^{n} x_i r_i = \bar{r}, \sum_{i=1}^{n} x_i = 1, x_i \geq 0\ \}$.
The Lagrange function for problem (2.7) is

$$L(x, \lambda, \mu) = \langle r, x\rangle - \frac{1}{2}\langle Cx, x\rangle + \lambda_1\left(\sum_{i=1}^{n} x_i - 1\right) + \lambda_2\left(\sum_{i=1}^{n} r_i x_i - r\right) - \sum_{i=1}^{n} \mu_i x_j.$$

Since problem (2.7) has non-negative constraints, in order to solve the problem, we need to use Conditional Gradient Method. Algorithm of this method is the following.

Algorithm CGM

Step 1: Choose an arbitrary point $x^0 \in S$, and set $k = 0$. Compute the gradient $f'\left(x^k\right)$.

Step 2: Solve the following linear programming sub-problem:

$$\max_{x \in S} \langle f'(x^k), x\rangle.$$

Let $\bar{x}^k$ be a solution to this problem

$$\langle f'(x^k), \bar{x}^k\rangle = \max_{x \in S} \langle f'(x^k), x\rangle.$$

Step 3: Compute a value of η_k as follows:

$$\eta_k = \langle f'(x^k), \bar{x}^k - x^k\rangle.$$

Step 4: If $\eta_k = 0$ then x^k is a solution. Otherwise, go to step 5.

Step 5: Construct a direction h^k as $h^k = \bar{x}^k - x^k$.

Step 6: Solve one-dimensional maximization problem:

$$\max_{0 \leq \alpha \leq 1} \varphi(\alpha) = f(x^k + \alpha h^k). \tag{2.8}$$

Let α_k be a solution to this problem:

$$\varphi(\alpha_k) = \max_{0 \leq \alpha \leq 1} \varphi(\alpha).$$

Step 7: Construct a point $z = x^k + \alpha_k h^k$

Step 8: Set $k := k + 1$ and $x^k := z$ and go to step 2.

Problem 2.8 can be solved analytically:

$$\varphi(\alpha) = f\left(x^k + \alpha h^k\right) = \langle r, x^k + \alpha h^k\rangle - \frac{1}{2}\langle C(x^k + \alpha h^k), x^k + \alpha h^k\rangle =$$
$$= \langle r, x^k\rangle + \alpha\langle r, x^k\rangle - \frac{1}{2}\langle Cx^k, x^k\rangle - \alpha\langle Cx^k, h^k\rangle - \frac{1}{2}\alpha^2\langle Ch^k, h^k\rangle.$$

Note that if $\langle Ch^k, h^k\rangle$ then $h^k = 0$ which implies $\bar{x}^k = x^k$ and $\eta_k = 0$. Thus, $x^* = x^k$ is a global solution to the problem.

If $\langle Ah^k, h^k \rangle$ then $\varphi(\alpha)$ is a quadratic concave function that reaches its maximum value at a point α: $\varphi'(\alpha) = 0$.
Hence we have

$$\langle r, h^k \rangle - \langle Cx^k, h^k \rangle - \alpha \langle Ch^k, h^k \rangle - \alpha \langle Ch^k, h^k \rangle = 0,$$
$$\alpha^* = \frac{\langle r - Cx^k, h^k \rangle}{\langle Ch^k, h^k \rangle} = \frac{\langle f'(x^k), h^k \rangle}{\langle Ch^k, h^k \rangle} \ or$$
$$\alpha^* = \frac{\langle \eta_k \rangle}{\langle Ch^k, h^k \rangle}.$$

Now we can choose α_k as follows:

$$\alpha_k = \begin{cases} \alpha^*, & 0 \le \alpha^* \le 1 \\ 1, & \alpha^* > 1 \end{cases}.$$

The convergence of the Algorithm is given by the following assertion.

Theorem 2.1. [5] *The sequence $\{x^k, k = 1, 2, \ldots\}$ generated by Algorithm-CGM converges to a global maximum point of the problem.*

$$\lim_{x^k \to \infty} x^k = x^* \tag{2.9}$$

where, $f(x^*) = \max_{x \in S} f(x)$.

3. The results

3.1. Numerical Experiment

For the numerical experiment, we used the last 6 years of price data [6] of 7 assets in Table 1. In light of the current economic downturn, we included the types of assets that have rising rates, such as a treasury bond, a volatility index, USD/JPY rate, a coal company stock, and three kinds of funds. The Conditional Gradient Algorithm was implemented in Matlab.

Table 1: Assets in the Portfolio.

Assets	U.S Treasury Yield, 10 years	Exchange rate for USD and JPY	CBOE Volatility Index	Whitehaven Coal Limited	Cambria Value and Momentum ETF	Catalyst/ Millburn Hedge Strategy Fund, Class C	LoCorr Market Trend Fund, Class A
Tickers	$\hat{}TNX, x_1^*$	$JPY = X, x_2^*$	$\hat{}VIX, x_3^*$	$WHITF, x_4^*$	$VAMO, x_5^*$	$MBXCX, x_6^*$	$LOTAX, x_7^*$

Covariance and expected rates of return matrixes are given as follows:

$$C = 10^{-4} * \begin{pmatrix} 12.95 & 0.13 & (6.38) & 0.21 & 1.02 & 1 & 0.02 \\ 0.13 & 0.24 & (0.12) & 0.07 & (0.01) & 0.02 & 0.01 \\ (6.38) & (0.12) & 68.74 & (2.44) & (4.7) & (4.54) & (1.96) \\ 0.21 & 0.07 & (2.44) & 11.27 & 0.47 & 0.31 & 0.22 \\ 1.02 & (0.01) & (4.7) & 0.47 & 1.71 & 0.59 & 0.28 \\ 1 & 0.02 & (4.54) & 0.31 & 0.59 & 0.86 & 0.32 \\ (0.02) & 0.01 & (1.96) & 0.22 & 0.28 & 0.32 & 0.74 \end{pmatrix},$$

$$r = \begin{pmatrix} 0.0005 \\ 0.0002 \\ 0.0005 \\ 0.0011 \\ 0.0001 \\ 0.0003 \\ 0.0002 \end{pmatrix}.$$

The maximum growth rate of Problem 1 is 0.07% with the following asset weights:

$$X = \begin{pmatrix} 27\% \\ -36\% \\ 9\% \\ 79\% \\ -100\% \\ 100\% \\ 20\% \end{pmatrix}.$$

The maximum growth rate of Problem 2 is 0.06% with the following asset weights:

$$X = \begin{pmatrix} 20.4\% \\ 0\% \\ 7.2\% \\ 71\% \\ 0\% \\ 1.5\% \\ 0\% \end{pmatrix}.$$

4. Conclusions

We have shown that the multi-objective optimization problem which consists of maximizing the return of the portfolio and minimizing the risk of the portfolio reduces to the problem of maximizing the growth rate of the portfolio. Numerical experiments have been done.

References

[1] D. G. Luenberger, "Investment Science," *Oxford University Press, Inc*, pp. 158-430, 2009.

[2] H. Markowitz, "Portfolio Selection," *The Journal of Finance*, Vol. 7, no. 1, Mar., pp. 77-91, 1952, doi: https://doi.org/10.1111/j.1540-6261.1952.tb01525.x.

[3] R. Enkhbat, "Optimization 5," *National University of Mongolia Press*, pp. 77-103, 2018.

[4] J. P. Aubin, "Optima and Equilibria," *University Paris-Dauphine*, pp. 108-110, 1998.

[5] J. Nocedal, and S. J. Wright, "Numerical Optimization," *Springer-Verlag New York, Inc*, 2006.

[6] *Yahoo Finance Website:*, https://finance.yahoo.com/.

[7] G. Gantigmaa, "Optimal Portfolio Selection of Currencies," *Journal of Institute of Mathematics and Digital Technology-3*, Vol. 3, pp. 29-35, 2021.

[8] M. Vaclavik, and J. Jablonsky, "Revisions of Modern Portfolio Theory Optimization Model," *Springer-Verlag*, Sep., pp. 474, 2011, doi: https://doi.org/10.1007/s10100-011-0227-2.

[9] Y. Zang, X. Li, and S. Guo, "Portfolio Selection Problems with Markowitz's mean-variance framework: a review of literature," *Fuzzy Optim Decis Making 17, Springer*, pp. 125-158, 2018, doi: https://doi.org/10.1007/s10700-017-9266-z.

[10] O. Mionel, and A. Moraru, "The Trend of International Risk Diversification," *Acta Universitatis Danubius Economica*, Vol. 9, no. 5, pp. 235-246.

[11] Ya. Bazarsad, and R. Enkhbat, "Probability Theory and Mathematical Statistics, Institute of Information and Communication Technology," *Mongolian University of Science and Technology*, pp. 31-48, 2008.

[12] Ya. Bazarsad, M. Banzragch, V. Batzorig, P. Dorjnaran, U. Delgersaikhan, Ya. Lutbat, N. Munkhdalai, and R. Enkhbat, "Economic Methods and Models, Institute of Information and Communication Technology," *Mongolian University of Science and Technology*, pp. 80-85, 2007.

[13] L. Harris, "Trading and Exchange: Market Microstructure for Practitioners," *Oxford University Press*, pp. 41.

[14] G. Debreu, "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences of the United States of America*, Vol. 40, no. 7, Jul., 1954, doi: https://doi.org/10.1073/pnas.40.7.588.

[15] J. Cvitanic, V. Polimenis, and F. Zapatero, "Optimal Portfolio Allocation with Higher Moments," *Annals of Finance*, Vol. 4, Mar., 2007, doi: https://doi.org/10.1007/s10436-007-0071-5.

Багцын Өгөөжийн Өсөлт ба Парето Оновчлол

Рэнцэнгийн Энхбат, Ганлхагвын Гантигмаа*

Шинжлэх Ухааны Академи, Математик, тоон технологийн хүрээлэн, Улаанбаатар 13330, Монгол улс

**Холбоо барих зохиогч: gantigmaa_g@mas.ac.mn; ORCID:0000-0002-9604-9233*

Өгүүллийн мэдээлэл: Хүлээн авсан: 2022.09.15; Зөвшөөрөгдсөн: 2022.10.01; Нийтлэгдсэн: 2022.12.26

Хураангуй: Хөрөнгө оруулалтын шинжлэх ухаанд багцын оновчлол чухал үүрэг гүйцэтгэдэг. Бид энэхүү ажилд Марковицын сонгодог загварыг Паретогийн оновчлолтой холбон, багцын өгөөжийг нэмэгдүүлэх, эрсдлийг багасгахын тулд олон зорилтот оновчлолын бодлогыг авч үзлээ. Олон зорилтот оновчлолын Парето шийдүүд нь багцын хамгийн их өсөлтийг тодорхойлох асуудалд хариу өгдөг гэдгийг бид харууллаа. Тоон үр дүнг Матлаб ашиглан гаргасан.

Түлхүүр үгс: Марковицийн онол, Олон-зорилтот оптимизаци