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Refined L^2-decay estimate of solutions to a system of dissipative nonlinear Schrödinger equations

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Abstract. We study the Cauchy problem of a system of dissipative nonlinear Schrödinger equations. Our target is to obtain a time-decay estimate of the solutions in L^2 without size-restriction on the initial data. By imposing high regularity and rapid spatial decay on the initial data, we find that the L^2-norm of the solutions decays at the rate of $O((\log t)^{-2/5})$ as $t \to \infty$. It is a refined decay-estimate compared to former works.

Keywords: Nonlinear Schrödinger equation, L^2-decay estimate, scale-invariant weighted estimate.

1 Introduction

In this paper, we consider a Cauchy problem for a system of nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t \mathbf{u} + \frac{1}{2}\partial_x^2 \mathbf{u} = \mathbf{f}(\mathbf{u}) \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \end{cases} \tag{1}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}$ and the unknown variable is

$$\mathbf{u} = \mathbf{u}(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))^t \in \mathbb{C}^n$$

for $n \geq 1$ (the number of entries). The nonlinearity $\mathbf{f}(\mathbf{u})$ is assumed to be a $\mathbb{C}^n$-valued gauge-invariant cubic polynomial of u_j and $\overline{u}_j$ $(j = 1, 2, \cdots, n)$, i.e.,

$$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), \cdots, f_n(\mathbf{u}))^t \text{ with } f_j(\mathbf{u}) \in \mathbb{C} \ (j = 1, 2, \cdots, n), \tag{2}$$

$$\mathbf{f}(e^{i\theta}\mathbf{u}) = e^{i\theta}\mathbf{f}(\mathbf{u}) \text{ for any } \theta \in \mathbb{R} \text{ and } \mathbf{u} \in \mathbb{C}^n, \tag{3}$$

$$\mathbf{f}(\rho\mathbf{u}) = \rho^3\mathbf{f}(\mathbf{u}) \text{ for any } \rho > 0 \text{ and } \mathbf{u} \in \mathbb{C}^n. \tag{4}$$

The system of nonlinear Schrödinger equations arises in optical fiber engineering [1]. It describes the evolutions of light pulses (or electric fields) propagating through an optical fiber, and the solutions $u_1, u_2, \cdots, u_n$, or precisely the absolute values of the solutions, denote enveloping curves of quickly oscillating electric fields. In this model, each of $u_1, u_2, \cdots, u_n$ indicates an electric filed oscillating with distinguished frequency. These

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electric fields are interacted with each other under the nonlinearity. The model (1) contains nonlinear dissipative interaction as well.

The purpose of this paper is to show an L^2-decay estimate of the solution to (1) without assuming size restriction on the initial data $\mathbf{u}_0$. In order to remove the size-restriction on the initial data, a structural assumption on the nonlinearity is required, i.e., we assume that there exists some $\rho > 0$ such that, for any $\mathbf{p} = (p_1, p_2, \cdots, p_n)^t \in \mathbb{C}^n$,

$$\mathrm{Im}\left\{\overline{\mathbf{p}}^t \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})\mathbf{p} \pm \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{\mathbf{p}}\right)\right\} \leq -\rho \sum_{j=1}^{n} |u_j|^2 |p_j|^2, \tag{5}$$

where $\overline{\mathbf{p}} = (\overline{p_1}, \overline{p_2}, \cdots, \overline{p_n})^t$, $\partial \mathbf{f}/\partial \mathbf{u}$ (resp. $\partial \mathbf{f}/\partial \overline{\mathbf{u}}$) is a matrix, the jk-entry of which is given by $\partial f_j/\partial u_k$ (resp. $\partial f_j/\partial \overline{u_k}$). We call (5) *the strong dissipative condition* on the nonlinearity. We here want to see some examples of $\mathbf{f}(\mathbf{u})$ satisfying (2)–(5). In the case of $n = 1$, i.e., in the case of single equation, we have $\mathbf{f}(u) = \lambda |u|^2 u$ with $\lambda \in \mathbb{C}$, $\mathrm{Im}\lambda < 0$ and $|\mathrm{Re}\lambda| < \sqrt{3}|\mathrm{Im}\lambda|$ (or equivalently $2|\mathrm{Im}\lambda| - |\lambda| > 0$). In fact, it follows that, for $p \in \mathbb{C}$,

$$\begin{aligned}\mathrm{Im}\left\{\overline{p} \cdot \left(\frac{\partial \mathbf{f}}{\partial u}(u)p \pm \frac{\partial \mathbf{f}}{\partial \overline{u}}(u)\overline{p}\right)\right\} &= 2\mathrm{Im}\lambda |u|^2 |p|^2 \pm \mathrm{Im}(\lambda u^2 \overline{p}^2) \\ &\leq -(2|\mathrm{Im}\lambda| - |\lambda|)|u|^2 |p|^2,\end{aligned}$$

and we know that $\rho = 2|\mathrm{Im}\lambda| - |\lambda| > 0$ satisfies (5). In the case of $n = 2$, we have

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \lambda_{11}|u_1|^2 u_1 + \lambda_{12}|u_2|^2 u_1 \\ \lambda_{21}|u_1|^2 u_2 + \lambda_{22}|u_2|^2 u_2 \end{pmatrix}$$

with $\mathbf{u} = (u_1, u_2)^t$, $\lambda_{jk} \in \mathbb{C}$ $(j, k = 1, 2)$ and

$$\begin{aligned}&\mathrm{Im}\lambda_{jk} < 0, \\ &2|\mathrm{Im}\lambda_{11}| > |\lambda_{11}| + 2|\lambda_{12}| + |\lambda_{21}|, \\ &2|\mathrm{Im}\lambda_{22}| > |\lambda_{22}| + |\lambda_{12}| + 2|\lambda_{21}|.\end{aligned} \tag{6}$$

Actually it follows that

$$\begin{aligned}\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}} &= \begin{pmatrix} \frac{\partial \mathbf{f}(\mathbf{u})}{\partial u_1} & \frac{\partial \mathbf{f}(\mathbf{u})}{\partial u_2} \end{pmatrix} = \begin{pmatrix} 2\lambda_{11}|u_1|^2 + \lambda_{12}|u_2|^2 & \lambda_{12}\overline{u_2}u_1 \\ \lambda_{21}\overline{u_1}u_2 & \lambda_{21}|u_1|^2 + 2\lambda_{22}|u_2|^2 \end{pmatrix}, \\ \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{\mathbf{u}}} &= \begin{pmatrix} \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{u_1}} & \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{u_2}} \end{pmatrix} = \begin{pmatrix} \lambda_{11}u_1^2 & \lambda_{12}u_2 u_1 \\ \lambda_{21}u_1 u_2 & \lambda_{22}u_2^2 \end{pmatrix},\end{aligned}$$

and so, by Young's inequality $2|u_1||u_2||p_1||p_2| \leq |u_1|^2|p_1|^2 + |u_2|^2|p_2|^2$ and (6), we have

$$\begin{aligned}&\mathrm{Im}\left\{\overline{\mathbf{p}}^t \cdot \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})\mathbf{p} \pm \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{\mathbf{p}}\right)\right\} \\ &\leq -(2|\mathrm{Im}\lambda_{11}| - |\lambda_{11}| - 2|\lambda_{12}| - |\lambda_{21}|)|u_1|^2|p_1|^2 \\ &\quad - (2|\mathrm{Im}\lambda_{22}| - |\lambda_{22}| - |\lambda_{12}| - 2|\lambda_{21}|)|u_2|^2|p_2|^2.\end{aligned}$$

Therefore, by taking $\rho = \min\{2|\mathrm{Im}\lambda_{11}| - |\lambda_{11}| - 2|\lambda_{12}| - |\lambda_{21}|, 2|\mathrm{Im}\lambda_{22}| - |\lambda_{22}| - |\lambda_{12}| - 2|\lambda_{21}|\} > 0$, we see that the nonlinearity satisfies (5).

Let us next see the known researches on the dissipative and non-dissipative NLS's. We first consider the single case, i.e.,

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u \\ u(0,x) = u_0(x). \end{cases} \tag{7}$$

If $\mathrm{Im}\lambda = 0$ (non-dissipative case), Hayashi-Naumkin [7] proved that, under the smallness assumption on u_0 in some weighted Sobolev space, the solution to (7) decays like $O(t^{-1/2})$ as $t \to \infty$ in L^∞ and conserves L^2-norm – these properties are the same as the solution to the nonlinearity-free equation. However the asymptotic behavior of $u(t)$ contains a certain phase correction added to the free profile, i.e., the nonlinear effect is visible in phase. If $\mathrm{Im}\lambda < 0$ (dissipative case), Shimomura [27] proved that the small-data-solutions decay like $O(t^{-1/2}(\log t)^{-1/2})$ in L^∞ and decay like $o(1)$ in L^2 as $t \to \infty$ – the nonlinear dissipative effect is now visible in the additional logarithmic decay-rate as well as in the asymptotic leading term. Under some smallness assumptions on u_0, the effect caused by the nonlinear dissipation has been studied not only for (7) but also for NLS of super-critical and sub-critical power of nonlinearity (see [5, 11, 19]). In addition, the derivative type of nonlinearities was studied in [8].

The research on the dissipative NLS has been progressed to solving a problem of removing the size-restriction on u_0. Under the strong dissipative condition on the nonlinearity, i.e.,

$$\mathrm{Im}\lambda < 0, \quad |\mathrm{Re}\lambda| \le \sqrt{3}|\mathrm{Im}\lambda|, \tag{8}$$

Kita-Shimomura [18] obtained $\|u(t)\|_{L^\infty} = O(t^{-1/2}(\log t)^{-1/2})$ and $\|u(t)\|_{L^2} = o(1)$ as well as an asymptotic leading term of $u(t)$. In [18], the key to remove the smallness assumption is to show $\|Ju(t)\|_{L^2} \le \|xu_0\|_{L^2}$, where $J = x + it\partial_x$. The upper bound of $\|Ju(t)\|_{L^2}$ is derived by (8), which is the $n = 1$ version of (5). Thereafter Hayashi-Li-Naumkin [6] proved that, for $u_0 \in H^1$ with $xu_0 \in L^2$, the solution $u(t)$ to (7) satisfies an L^2-decay estimate such as

$$\|u(t)\|_{L^2} = O((\log t)^{-1/3}). \tag{9}$$

For the initial data of low regularity, see also [9, 10]. The first aim of this paper is to obtain a better-decay estimate compared to (9) for u_0 in more restricted function space. Cazenave-Naumkin [4] succeeded in removing the strong dissipative condition, and obtained decay estimate of the solutions in L^∞ and L^2 under the normal dissipative nonlinear structure (i.e., only $\mathrm{Im}\lambda < 0$). In their work, the initial data is assumed to be of the oscillating form such as

$$u_0(x) = e^{ibx^2} v_0(x) \tag{10}$$

with $b > 0$ sufficiently large. The idea of their proof is based on the pseudo-conformal transform of u, for which the effect of nonlinearity is well-extracted and the oscillating condition (10) works well. For sub-critical nonlinearities, see also [2, 3, 6, 12]. The second aims of this paper is to obtain an L^2-decay estimate without (10). Recently Sato [26] has proved that, under (8), the regularity of u_0 refines the L^2-decay rate. Precisely speaking, if $u_0 \in H^m$ and $xu_0 \in H^{m-1}$, then it holds that

$$\|u(t)\|_{L^2} = O((\log t)^{-1/2+1/(4m+2)}). \tag{11}$$

The estimate (11) suggests that, by taking the regularity m large, the decay rate of $\|u(t)\|_{L^2}$ is refined. The third aim of this paper is to show a similar L^2-decay estimate for the system of nonlinear Schrödinger equations, with $m = 2$ and somewhat stronger spatial-decay condition.

We next introduce some known results on the systems of NLS. In early researches, the nonlinear dissipation was not considered. Therefore the dynamics of the solutions was considered only for small-sized data. This is because the estimate of $J\mathbf{u}(t)$, which is the key to control the error terms, indicated that the growth-rate of $\|J\mathbf{u}(t)\|_{L^2}$ depended on the size of initial data. Therefore, to guarantee the rapid decay of the error terms, one needs restricting the size of data. Anyway, under the smallness assumption of the data, Katayama-Sakoda [13] considered the case where the nonlinearity contains derivatives of unknown variables, and proved the asymptotic behavior of solutions by imposing a certain structural condition on the nonlinearity – it has something to do with the method of Lyapunov function. In most cases of the systems, it becomes more difficult than in the case of single equation, to detect a concrete form of the asymptotic behavior term of the solution, since the well-known technique of gauge-transform of the unknown variables does not work so well. However, in [13], the asymptotic leading term is described by a function satisfying the ordinary differential equation associated with the system of NLS.

System of NLS features solutions of complicated asymptotic behaviors, which are apparently explained by the ordinary differential equations associated with the NLS system. Masaki [23] classified types of the 2-component systems of the ordinary differential equations associated with NLS systems. His work suggests which nonlinearities are essential for the asymptotic dynamics of nonlinear dispersive equations. For some nonlinearities included in his classification, Masaki-Segata-Uriya [24] and Kita-Masaki-Segata-Uriya [15] considered coupled NLS, for which the solutions decay more slowly than those of free Schrödinger equations.

As for the works considering dissipative structures in system, Kim [14] considered the case of distinct masses (i.e., coefficients of ∂_x^2). He proved that the L^∞-norm of the solutions decays like $O(t^{-1/2}(\log t)^{-1/2})$. Li-Nishii-Sagawa-Sunagawa [20] handled another kind of system, and proved the existence of solutions indicating that one component rapidly decays but the other is asymptotically free. The works introduced above suggest that solutions to the systems of NLS exhibit complicated asymptotic profiles which are not simply expected by the single NLS. For the case of derivative nonlinearities, see also [21, 22, 25].

As we have seen above, all of the works on NLS systems imposed the size-restriction on the data. However, under the strong dissipative condition (5), Kita-Nakamura-Sagawa [17] proved that, if $\mathbf{u}_0 \in H^1$ and $x\mathbf{u}_0 \in L^2$ without size restriction, there exists a unique global solution to (1) such that

$$\mathbf{u} \in C([0,\infty); H^1) \cap C^1([0,\infty); H^{-1}), \quad x\mathbf{u} \in C([0,\infty); L^2).$$

Furthermore the L^∞-norm decays like

$$\|\mathbf{u}(t)\|_{L^\infty} = O(t^{-1/2}(\log t)^{-1/2}). \tag{12}$$

According to [16], the L^∞-decay rate $O(t^{-1/2}(\log t)^{-1/2})$ is optimal in the sense that, if $\|\mathbf{u}(t)\|_{L^\infty}$ decays more rapidly than $t^{-1/2}(\log t)^{-1/2}$, then $u \equiv 0$. In this paper, we are interested in a decay estimate of the solution in L^2 (not in L^∞). It is formulated as follows.

Theorem 1 **(L^2-decay estimate).** *Let the nonlinearity* $\mathbf{f}(\mathbf{u})$ *satisfy* (2) – (5) *and* $\mathbf{u}_0 \in H^2$, $x^2\mathbf{u}_0 \in L^2$ *without size-restriction. Then there exists a unique solution* $\mathbf{u}(t,x)$ *to* (1) *such that*

$$\mathbf{u} \in C([0,\infty); H^2) \cap C^1([0,\infty); L^2) \quad \textit{and} \quad x^2\mathbf{u} \in C([0,\infty); L^2). \tag{13}$$

Furthermore $\mathbf{u}(t,x)$ *satisfies*

$$\|\mathbf{u}(t,\cdot)\|_{L^2} = O((\log t)^{-2/5}) \tag{14}$$

as $t \to \infty$.

We emphasize that the decay rate in (14) is refined, compared to (9). We did not assume the oscillating form on the initial data as in (10). Note that the decay rate is similar to that obtained in [26], under the strong dissipative condition (5) for the system.

We want to close this section by introducing some notation. The Lebesgue space L^q $(1 \le q \le \infty)$ for $\mathbb{C}^n$-valued functions is defined by

$$L^q = \{\mathbf{u}(x) = (u_1(x), u_2(x), \cdots, u_n(x))^t \ ; \ \|\mathbf{u}\|_{L^q} < \infty\},$$

where $\|\mathbf{u}\|_{L^q} = \left(\int_{\mathbb{R}} |\mathbf{u}(x)|^q \, dx\right)^{1/q}$ with $|\mathbf{u}| = (\sum_{j=1}^n |u_j|^2)^{1/2}$ if $1 \le q < \infty$, and $\|\mathbf{u}\|_{L^\infty} = \text{ess.sup}_{x\in\mathbb{R}}|\mathbf{u}(x)|$. The Sobolev space H^m is endowed with

$$\|\mathbf{u}\|_{H^m} = \left(\sum_{k=0}^m \|\partial_x^k \mathbf{u}\|_{L^2}^2\right)^{1/2}.$$

The solution operator of the linear Schrödinger equation is denoted by $U(t)$ or $\exp(it\partial_x^2/2)$. Then $U(t)$ is factorized like

$$U(t) = MDFM,$$

where $M = e^{ix^2/2t}$ is the multiplication operator, $Df(x) = (it)^{-1/2} f(x/t)$ and F is the Fourier transform. The operator $J(t)$ is defined by

$$J(t) = U(t)\, xU(-t) = Mit\partial_x \overline{M} = x + it\partial_x.$$

Note that $J(t)$ is the infinitesimal generator of the Galilei transformation, and it commutes with $i\partial_t + \frac{1}{2}\partial_x^2$.

The paper is organized as follows. In Section 2, we will collect necessary estimates of $\|\mathbf{u}(t)\|_{L^2}$, $\|\partial_x^k \mathbf{u}(t)\|_{L^2}$ and $\|J(t)^k \mathbf{u}(t)\|_{L^2}$ $(k = 1, 2)$. A scale-invariant weighted estimate will be also proved. By making use of these estimates, we will prove Theorem 1.

2 Preliminary Estimates and Proof of Theorem 1

The next lemma plays an important role in our proof.

Lemma 1. *Let* $\mathbf{u}_0 \in H^2$ *and* $x^2\mathbf{u}_0 \in L^2$. *We assume* (2)–(5) *for the nonlinearity. Then, there exists some positive constant* C *depending on* $\mathbf{u}_0$ *such that, for* $t \geq 2$, *the solution to (1) satisfies*

$$\|\mathbf{u}(t)\|_{L^2} \leq C, \tag{15}$$

$$\|J(t)\mathbf{u}(t)\|_{L^2} \leq C, \tag{16}$$

$$\|\partial_x \mathbf{u}(t)\|_{L^2} \leq C, \tag{17}$$

$$\|J(t)^2\mathbf{u}(t)\|_{L^2} \leq C(\log t)^2, \tag{18}$$

$$\|\partial_x^2 \mathbf{u}(t)\|_{L^2} \leq C. \tag{19}$$

Proof (**Lemma 1).** We will proceed in formal way, which means that we will not discuss the differentiability of $J(t)\mathbf{u}$, $J(t)^2\mathbf{u}$ and $\partial_t\mathbf{u}$ – It is justified by the cut-off and regularization technique. By (3), we see that

$$e^{i\theta}\mathbf{f}(\mathbf{u}) = \mathbf{f}(e^{i\theta}\mathbf{u}).$$

Differentiating the both hand sides with respect to θ and next substituting $\theta = 0$, we have

$$\mathbf{f}(\mathbf{u}) = \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}}\mathbf{u} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{\mathbf{u}}}\overline{\mathbf{u}}.$$

Then (5) leads to

$$\frac{d\|\mathbf{u}\|_{L^2}^2}{dt} = \mathrm{Im} \int \left\{ \overline{\mathbf{u}}^t \left(\frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}}\mathbf{u} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \overline{\mathbf{u}}}\overline{\mathbf{u}} \right) \right\} dx \leq 0. \tag{20}$$

Hence (15) follows. We next prove (16). By $J(t) = Mit\partial_x\overline{M}$ and the gauge-invariance of $\mathbf{f}(\mathbf{u})$ (described in (3)), we see that

$$\begin{aligned} J(t)\mathbf{f}(\mathbf{u}) &= \sum_{k=1}^{n} \frac{\partial \mathbf{f}}{\partial u_k}(\mathbf{u})J(t)u_k - \sum_{k=1}^{n} \frac{\partial \mathbf{f}}{\partial \bar{u}_k}(\mathbf{u})\overline{J(t)u_k} \\ &= \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})J(t)\mathbf{u} - \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{J(t)\mathbf{u}}. \end{aligned}$$

The strong dissipative condition (5) is utilized to estimate $\mathrm{Im}\left\{(\overline{J(t)\mathbf{u}})^t \cdot J(t)\mathbf{f}(\mathbf{u})\right\}$. In fact, by (5) with $\mathbf{p} = J(t)\mathbf{u}$, the solution $\mathbf{u}$ satisfies

$$\begin{aligned} \frac{d\|J(t)\mathbf{u}\|_{L^2}^2}{dt} &= \int \mathrm{Im}\left\{\overline{J(t)\mathbf{u}} \cdot \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})J(t)\mathbf{u} - \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{J(t)\mathbf{u}}\right)\right\}\, dx \\ &\le -\rho_1 \int \sum_{j=1}^{n} |u_j|^2 |J(t)u_j|^2\, dx \\ &\le 0. \end{aligned}$$

Therefore, for $t \ge 0$, we have (16). The estimate (17) follows by the chain rule such as

$$\partial_x \mathbf{f}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})\partial_x \mathbf{u} + \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{\partial_x \mathbf{u}}$$

and by the analogy for (16). To prove (18), note that

$$\begin{aligned} J(t)^2\mathbf{f}(\mathbf{u}) = &\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u})J(t)^2\mathbf{u} - \frac{\partial \mathbf{f}}{\partial \overline{\mathbf{u}}}(\mathbf{u})\overline{J(t)^2\mathbf{u}} \\ &+ \sum_{1\le k,\ell\le n} \frac{\partial^2 \mathbf{f}}{\partial u_\ell \partial u_k}(\mathbf{u})J(t)u_\ell J(t)u_k - 2\sum_{1\le k,\ell\le n} \frac{\partial^2 \mathbf{f}}{\partial \overline{u_\ell} \partial u_k}(\mathbf{u})\overline{J(t)u_\ell} J(t)u_k \\ &+ \sum_{1\le k,\ell\le n} \frac{\partial^2 \mathbf{f}}{\partial \overline{u_\ell} \partial \overline{u_k}}(\mathbf{u})\overline{J(t)u_\ell J(t)u_k}. \end{aligned}$$

Then, by (5) with $\mathbf{p} = J(t)^2\mathbf{u}$, there exists some positive constant C such that

$$\frac{d\|J(t)^2\mathbf{u}\|_{L^2}^2}{dt} \le C\|\mathbf{u}\|_{L^\infty}\|J(t)\mathbf{u}\|_{L^\infty}\|J(t)\mathbf{u}\|_{L^2}\|J(t)^2\mathbf{u}\|_{L^2}. \tag{21}$$

Apply Gagliardo-Nirenberg's inequality to $\|\mathbf{u}\|_{L^\infty}$ and $\|J(t)\mathbf{u}\|_{L^\infty}$. Then we have

$$\begin{aligned} \|\mathbf{u}\|_{L^\infty} &= \|M\mathbf{u}\|_{L^\infty} \\ &\le C\|M\mathbf{u}\|_{L^2}^{1/2}\|\partial_x M\mathbf{u}\|_{L^2}^{1/2} \\ &= Ct^{-1/2}\|\mathbf{u}\|_{L^2}^{1/2}\|J(t)\mathbf{u}\|_{L^2}^{1/2} \end{aligned}$$

and

$$\begin{aligned} \|J(t)\mathbf{u}\|_{L^\infty} &= \|MJ(t)\mathbf{u}\|_{L^\infty} \\ &\le C\|MJ(t)\mathbf{u}\|_{L^2}^{1/2}\|\partial_x MJ(t)\mathbf{u}\|_{L^2}^{1/2} \\ &= Ct^{-1/2}\|J(t)\mathbf{u}\|_{L^2}^{1/2}\|J(t)^2\mathbf{u}\|_{L^2}^{1/2}. \end{aligned}$$

By (15) and (16), we see that

$$\|\mathbf{u}\|_{L^\infty} \leq Ct^{-1/2} \quad \text{and} \quad \|J(t)\mathbf{u}\|_{L^\infty} \leq Ct^{-1/2}\|J(t)^2\mathbf{u}\|_{L^2}^{1/2}. \tag{22}$$

Applying (22) to (21) , we have

$$\frac{d\|J(t)^2\mathbf{u}\|_{L^2}^2}{dt} \leq Ct^{-1}\|J(t)^2\mathbf{u}\|_{L^2}^{3/2}.$$

Then Gronwall's inequality leads to

$$\|J(t)^2\mathbf{u}\|_{L^2} \leq C(\log t)^2,$$

and (18) is proved. We are next going to prove (19). We first consider the estimate of $\|\partial_t u\|_{L^2}$. By applying ∂_t to (1) and making use of the strong dissipative condition (5) with $\mathbf{p} = \partial_t\mathbf{u}$, it follows that

$$\begin{aligned}\frac{d\|\partial_t\mathbf{u}\|_{L^2}^2}{dt} &= \int \operatorname{Im}\left\{\overline{\partial_t\mathbf{u}} \cdot \left(\frac{\partial\mathbf{f}}{\partial\mathbf{u}}(\mathbf{u})\partial_t\mathbf{u} - \frac{\partial\mathbf{f}}{\partial\overline{\mathbf{u}}}(\mathbf{u})\overline{\partial_t\mathbf{u}}\right)\right\} dx \\ &\leq -\rho_1 \int \sum_{j=1}^n |u_j|^2|\partial_t u_j|^2 \, dx \\ &\leq 0.\end{aligned}$$

Hence we have

$$\begin{aligned}\|\partial_t\mathbf{u}\|_{L^2} &\leq \|\partial_t\mathbf{u}(t)|_{t=0}\|_{L^2} \\ &= \| -\frac{1}{2}\partial_x^2\mathbf{u}_0 + \mathbf{f}(\mathbf{u}_0)\|_{L^2} \\ &\leq C(\|\mathbf{u}_0\|_{H^2} + \|\mathbf{u}_0\|_{L^2}^3).\end{aligned} \tag{23}$$

By (1), we find that

$$\begin{aligned}\|\partial_x^2\mathbf{u}\|_{L^2} &\leq 2\|\partial_t\mathbf{u}\|_{L^2} + 2\|\mathbf{f}(\mathbf{u})\|_{L^2} \\ &\leq 2\|\partial_t\mathbf{u}\|_{L^2} + C\|\mathbf{u}(t)\|_{H^1}^3.\end{aligned}$$

Then (23), (15) and (17) yield (19).

To determine the L^2-decay rate of the solution, the next scale-invariant weighted estimate will be required.

Lemma 2. *(Scale-invariant weighted estimate) Let* $\mathbf{u} \in L^2$ *and* $x^2\mathbf{u} \in L^2$*. Then there exists some positive constant* C *such that*

$$\|\mathbf{u}\|_{L^1} \leq C\|x^2\mathbf{u}\|_{L^2}^{1/4}\|\mathbf{u}\|_{L^2}^{3/4}. \tag{24}$$

Proof (**Lemma 2).** *Proof*. By Cauchy-Schwarz' inequality, it is easy to see that

$$\begin{aligned}\|\mathbf{u}\|_{L^1} &\le C\|(1+|x|^2)\mathbf{u}\|_{L^2}\\ &\le C\left(\|\mathbf{u}\|_{L^2}+\|x^2\mathbf{u}\|_{L^2}\right).\end{aligned} \tag{25}$$

We here take $\mathbf{u}_\eta(x) = \eta\mathbf{u}(\eta x)$ for $\eta > 0$. Note that $\|\mathbf{u}_\eta\|_{L^1} = \|\mathbf{u}\|_{L^1}$. Then (25) leads to

$$\|\mathbf{u}\|_{L^1} \le C\left(\eta^{1/2}\|\mathbf{u}\|_{L^2} + \eta^{-3/2}\|x^2\mathbf{u}\|_{L^2}\right). \tag{26}$$

By taking $\eta = \sqrt{3}\|x^2\mathbf{u}\|_{L^2}^{1/2}/\|\mathbf{u}\|_{L^2}^{1/2}$, the right hand side of (26) is minimized, and we have

$$\|\mathbf{u}\|_{L^1} \le C\left(\frac{4\cdot\sqrt[4]{3}}{3}\|x^2\mathbf{u}\|_{L^2}^{1/4}\|\mathbf{u}\|_{L^2}^{1/4}\right). \tag{27}$$

It completes the proof.

Now we are in the position to prove the L^2-decay estimate.

Proof (**Theorem 1).** The local well-posedness of the solution is easily proved by the contraction mapping principle in the function space:

$$X_T = \{\mathbf{u};\ \sup_{0\le t\le T}(\|\mathbf{u}(t)\|_{H^2} + \|J(t)^2\mathbf{u}(t)\|_{L^2}) < \infty\}.$$

Then Lemma 1 guarantees the global existence of the solution, and (13) follows. By the identity (20) and strong dissipative condition (5), we have

$$\begin{aligned}\frac{d\|\mathbf{u}\|_{L^2}^2}{dt} &\le -\rho_1\sum_{j=1}^n\int|u_j|^4\,dx\\ &\le -\frac{\rho_1}{n}\int|\mathbf{u}|^4\,dx\\ &= -\frac{\rho_1}{n}\|\mathbf{u}\|_{L^4}^4,\end{aligned}$$

where Cauchy-Schwarz' inequality: $n\sum_{j=1}^n|u_j|^4 \ge |\mathbf{u}|^4$ was applied. We next apply Hölder's inequality: $\|\mathbf{u}\|_{L^2} \le \|\mathbf{u}\|_{L^1}^{1/3}\|\mathbf{u}\|_{L^4}^{2/3}$, and we have

$$\frac{d\|\mathbf{u}\|_{L^2}^2}{dt} \le -\frac{\rho_1}{n}\cdot\frac{\|\mathbf{u}\|_{L^2}^6}{\|\mathbf{u}\|_{L^1}^2}.$$

Applying Lemma 2 to the divisor on the right hand side, we see that

$$\frac{d\|\mathbf{u}\|_{L^2}^2}{dt} \le -\frac{\rho_1}{n}\cdot\frac{\|\mathbf{u}\|_{L^2}^{9/2}}{\|x^2\mathbf{u}\|_{L^2}^{1/2}}. \tag{28}$$

Since $x^2 = (J(t) - it\partial_x)^2$, the divisor is estimated as

$$\|x^2\mathbf{u}\|_{L^2} \le \|J(t)^2\mathbf{u}\|_{L^2} + t\|J(t)\partial_x\mathbf{u}\|_{L^2} + t\|\partial_x J(t)\mathbf{u}\|_{L^2} + t^2\|\partial_x^2\mathbf{u}\|_{L^2}. \tag{29}$$

Note that the self-adjointness of $J(t)$ leads to

$$\begin{aligned}\|J(t)\partial_x\mathbf{u}\|_{L^2}^2 &= \int J(t)\partial_x\mathbf{u}\cdot\overline{J(t)\partial_x\mathbf{u}}\,dx\\ &= \int \partial_x\mathbf{u}\cdot\overline{J(t)^2\partial_x\mathbf{u}}\,dx\\ &= -\int \partial_x^2\mathbf{u}\cdot\overline{J(t)^2\mathbf{u}}\,dx - 2\int \partial_x\mathbf{u}\cdot\overline{J(t)\mathbf{u}}\,dx\\ &\le \|\partial_x^2\mathbf{u}\|_{L^2}\|J(t)^2\mathbf{u}\|_{L^2} + 2\|\partial_x\mathbf{u}\|_{L^2}\|J(t)\mathbf{u}\|_{L^2},\end{aligned}$$

and similarly

$$\|\partial_x J(t)\mathbf{u}\|_{L^2}^2 \le \|\partial_x^2\mathbf{u}\|_{L^2}\|J(t)^2\mathbf{u}\|_{L^2} + 2\|\partial_x\mathbf{u}\|_{L^2}\|J(t)\mathbf{u}\|_{L^2}.$$

Then, by applying Young's inequality, (29) leads to

$$\begin{aligned}\|x^2\mathbf{u}\|_{L^2} \le C(1+t)\|J(t)^2\mathbf{u}\|_{L^2} + C(t+t^2)\|\partial_x^2\mathbf{u}\|_{L^2}\\ + Ct(\|\partial_x\mathbf{u}\|_{L^2} + \|J(t)\mathbf{u}\|_{L^2}).\end{aligned}$$

By Lemma 1, it turns out to be

$$\|x^2\mathbf{u}(t)\|_{L^2} \le C(1+t^2). \tag{30}$$

Applying (30) to (28), we see that

$$\frac{d\|\mathbf{u}\|_{L^2}^2}{dt} \le -C(1+t)^{-1}\|\mathbf{u}\|_{L^2}^{9/2},$$

or, equivalently,

$$\frac{d\|\mathbf{u}\|_{L^2}^{-5/2}}{dt} \ge 5C(1+t)^{-1}. \tag{31}$$

Integrating (31) from 0 to t, we see that

$$\|\mathbf{u}\|_{L^2}^{-5/2} \ge \|\mathbf{u}_0\|_{L^2}^{-5/2} + 5C\log(1+t),$$

and hence it holds that

$$\|\mathbf{u}\|_{L^2} \le C(\log(1+t))^{-2/5}.$$

It completes the proof of Theorem 1.

3 Discussion

In Theorem 1, we showed an L^2-decay estimate of the solution to the system of dissipative nonlinear Schrödinger equations (1). A remarking point is that a smallness assumption is not imposed on the initial data. Thanks to assuming a stronger regularity condition on the initial data, the decay rate is refined compared to Hayashi-Li-Naumkin's estimate [6]. In L^2, we can see not only the decay of hight of the wave but also the propagation speed of the wave. Since the dispersive effect means that the component of high frequency propagates in rapid speed, the regularity of the data will make the wave slowly propagate and the nonlinear dissipation largely affects the L^2-decay.

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